Improved Mixed-Boundary Component-Mode Representation for Structural Dynamic Analysis

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Main Reference

This presentation is a brief summary of the following technical paper:

Majed, A.[1], Henkel, E. E.[2], and Wilson, C.[3], "Improved Method of Mixed-Boundary Component-Mode Representation for Structural Dynamic Analysis," submitted for publication to *AIAA Journal of Spacecraft and Rockets*.

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Advantages

The subject Residual Flexibility Mixed-Boundary (*RFMB*) method has two compelling advantages over the other mixed-boundary methods

1. Analytical Advantage
   - Accurate for the entire range of component boundary representations, i.e., from all-fixed- to mixed- to all-free-boundaries

2. Test/Analysis Correlation Advantage
   - All parameters for the *RFMB* generalized stiffness matrix can be directly derived from component test, resulting in a more rigorous definition of test/analysis correlation
     – Not true for any other mixed-boundary method

Introduction

The challenge in Component-Mode Synthesis (CMS) is to develop specific variants with desired properties relative to both analysis and test

- Numerous technical papers written on "generalizations" since Hintz's 1975 paper [6] without ever deriving a specific, useful variant

Since the all fixed- and all free-boundary methods dictate the boundary conditions on the normal modes to be either all-fixed- or all-free-boundary, only the mixed-boundary approach offers the potential for developing a variant with a set of desired properties relative to both analysis and test

Specific Mixed-boundary (*MB*) variants:

- Hintz Mixed-Boundary (*HMB*) method
- Modified Hintz Mixed-Boundary (*MHMB*) method
- Residual Flexibility Mixed-Boundary (*RFMB*) method
- Residual Flexibility Mixed-Boundary Variant (*RFMB-V*) method

Desired Characteristics

- Accuracy: the method should yield an accurate representation of the component dynamics for <u>all</u> component boundary conditions including all-fixed and all-free
  - The robust and time-tested methods of Craig-Bampton (for all-fixed-boundary) and Rubin (for all-free-boundary) will be used to measure the accuracy of the various mixed-boundary method for the bounding cases of all-fixed- and all-free- boundary cases
- Explicit inclusion of physical boundary coordinates: the method should conform to simple direct stiffness CMS (component coupling) and not require any special coupling procedures
- Component independence: the method should be independent of other components' stiffness and mass boundary data
- Statically complete: the method should give the exact static solution for the component
- Test compatibility: the method's stiffness representation should be directly derivable from the component test, i.e., test/analysis correlation for all individual parameters comprising the component generalized stiffness matrix should be possible

HMB Method [6]

Utilizes constraint modes for fixed- and free-boundary coupling, inertia-relief modes, and normal modes

As derived in Hintz's paper, requires special coupling procedures

- Can be overcome with additional mathematics

For the all-free-boundary case, *HMB* method spans the same subspace as Rubin's method; i.e., methods are <u>equivalent</u>

HMB coordinate transformation

$$\begin{Bmatrix} x_b \\ x_c \\ x_i \end{Bmatrix} = \begin{bmatrix} \backslash I_{b} \backslash & 0 & 0 & 0 \\ 0 & \backslash I_{c} \backslash & 0 & 0 \\ \psi_{ib}^{C} & \psi_{ic}^{C} & \psi_{ir}^{I} & \phi_{ik}^{N} - \psi_{ic}^{C}\,\phi_{ck}^{N} \end{bmatrix} \begin{Bmatrix} x_b \\ x_c \\ q_r \\ q_k \end{Bmatrix}$$

HMB Method – Issues

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Analytical

- For the all-fixed-boundary case, Craig-Bampton results in superior convergence for higher modes (retains more normal modes for the same number of system dofs)
  - e.g., from classic Benfield truss example problem in Hintz's paper (12 dofs out of 60 dofs case), *HMB* results in a **10.3%** frequency error for the fifth mode compared to **0.19%** for Craig-Bampton
  - *HMB* gives superior convergence for the lower (fundamental) modes (*HMB* inertia-relief modes)

Test/Analysis Correlation

- *HMB* generalized stiffness matrix contains cross-coupling terms that can **not** be directly validated from component testing

MHMB Method

Popular variation of *HMB* which utilizes constraint modes for fixed- and free-boundary coupling, and normal modes

For the all-fixed-boundary case, *MHMB* is exactly equal to Craig-Bampton

For the mixed-boundary case (i.e., fixed-boundary sufficient to prevent rigid-body motion), *MHMB* spans the same subspace as *RFMB* and *RFMB*-V

- Constraint modes and total flexibility modes span the same subspace

MHMB coordinate transformation

$$\begin{Bmatrix} x_b \\ x_c \\ x_i \end{Bmatrix} = \begin{bmatrix} \lceil I_b \rfloor & 0 & 0 \\ 0 & \lceil I_c \rfloor & 0 \\ \psi_{ib}^{C} & \psi_{ic}^{C} & \phi_{ik}^{N} - \psi_{ic}^{C} \phi_{ck}^{N} \end{bmatrix} \begin{Bmatrix} x_b \\ x_c \\ q_k \end{Bmatrix}$$

MHMB Method – Issues

Analytical

- For the all-free-boundary case, *MHMB* is not equivalent to Rubin's method and demonstrates slow convergence and appreciable frequency errors
  - From classic Benfield truss example problem (shown later in this presentation), *MHMB* results in a **9.2%** frequency error for the third mode compared to **0.01%** for Rubin
    - Constraint modes and free-boundary normal modes alone do not span the same subspace as residual flexibility and free-boundary normal modes

Test/Analysis Correlation

- As is the case with the *HMB* method, *MHMB* generalized stiffness contains cross-coupling terms that can **not** be directly validated from component test

RFMB Method

The subject *RFMB* method involves three mutually linearly independent mode sets:

- Constraint modes representing the fixed-boundary
- Residual flexibility representing the free-boundary, and
- A truncated set of normal modes relative to the fixed-boundary

RFMB incorporates residual flexibility to "add" free-boundary coordinates to a Craig-Bampton without changing normal mode boundary conditions

- As such, *RFMB* is exactly equal to Craig-Bampton for the all-fixed-boundary case
- As such, *RFMB* is exactly equal to Rubin for the all-free-boundary case

RFMB generalized stiffness matrix is block-diagonal (no cross-coupling terms) which can be directly validated from component test (see test/analysis section)

RFMB Method – Cont'd

RFMB coordinate transformation

$$\begin{Bmatrix} x_b \\ x_c \\ x_i \end{Bmatrix} = \begin{bmatrix} \backslash I_{b}\backslash & 0 & 0 \\ 0 & \backslash I_{c}\backslash & 0 \\ \psi_{ib}^{C} - g_{ic}^{R} g_{cc}^{R\,-1} \psi_{cb}^{C} & g_{ic}^{R} g_{cc}^{R\,-1} & \phi_{ik}^{N} - g_{ic}^{R} g_{cc}^{R\,-1} \phi_{ck}^{N} \end{bmatrix} \begin{Bmatrix} x_b \\ x_c \\ q_k \end{Bmatrix}$$

RFMB

$$\begin{Bmatrix} x_b \\ x_i \end{Bmatrix} = \begin{bmatrix} \backslash I_{b}\backslash & 0 \\ \psi_{ib}^{C} & \phi_{ik}^{N} \end{bmatrix} \begin{Bmatrix} x_b \\ q_k \end{Bmatrix}$$

Craig-Bampton

(*RFMB* with all b-set)

$$\begin{Bmatrix} x_c \\ x_i \end{Bmatrix} = \begin{bmatrix} \backslash I_{c}\backslash & 0 \\ g_{ic}^{R} g_{cc}^{R\,-1} & \phi_{ik}^{N} - g_{ic}^{R} g_{cc}^{R\,-1} \phi_{ck}^{N} \end{bmatrix} \begin{Bmatrix} x_c \\ q_k \end{Bmatrix}$$

Rubin

(*RFMB* with all c-set)

RFMB-V Method [8]

RFMB-V is an equivalent variant to *RFMB* that incorporates total flexibility instead of residual flexibility

Issues

- *RFMB-V* generalized stiffness matrix involves cross-coupling terms that can **<u>not</u>** be directly validated from component test

RFMB-V coordinate transformation

$$\begin{Bmatrix} x_b \\ x_c \\ x_i \end{Bmatrix} = \begin{bmatrix} \lceil I_b \rfloor & 0 & 0 \\ 0 & \lceil I_c \rfloor & 0 \\ \psi_{ib}^C - g_{ic}^T {g_{cc}^T}^{-1} \psi_{cb}^C & g_{ic}^T {g_{cc}^T}^{-1} & \phi_{ik}^N - g_{ic}^T {g_{cc}^T}^{-1} \phi_{ck}^N \end{bmatrix} \begin{Bmatrix} x_b \\ x_c \\ q_k \end{Bmatrix}$$

Test/Analysis Correlation

The ability to validate the generalized stiffness parameters directly from test leads to a more rigorous test/analysis correlation, going well beyond the standard correlation practice of frequencies and normal modes

- *RFMB* generalized stiffness matrix parameters can be directly validated from component testing
  - Not true for any other mixed-boundary method
- *RFMB* allows the direct validation of the component interface flexibility dofs from modal test data

Martinez and Gregory [11] conclude that methods involving residual flexibility are best suited for direct determination of component parameters from test

Time-tested, practical techniques by Rubin [5], Lamontia [10], and Klosterman [12] are readily available for determining residual flexibility directly from component modal test data

Test/Analysis Correlation – Cont'd

***RFMB q1* Space**

$$x = \psi_b^C x_b + g_c^R q^R + \phi_k^N q^N \qquad \text{or} \qquad x = T_1 \; q_1$$

***RFMB q2* Space**

$$\begin{aligned} x_b &= x_b, \\ q^R &= -g_{cc}^{R^{-1}} (\psi_{cb}^C \; x_b - x_c + \phi_{ck}^N \; q^N), \\ q^N &= q^N \end{aligned} \qquad \text{or} \qquad q_1 = T_2 \; q_2$$

***RFMB* coordinate transformation**

$$x = T_1 \; T_2 \; q_2$$

$$\begin{Bmatrix} x_b \\ x_c \\ x_i \end{Bmatrix} = \begin{bmatrix} {}^{\backslash}I_{b\backslash} & 0 & 0 \\ 0 & {}^{\backslash}I_{c\backslash} & 0 \\ \psi_{ib}^C - g_{ic}^R g_{cc}^{R^{-1}} \psi_{cb}^C & g_{ic}^R g_{cc}^{R^{-1}} & \phi_{ik}^N - g_{ic}^R g_{cc}^{R^{-1}} \phi_{ck}^N \end{bmatrix} \begin{Bmatrix} x_b \\ x_c \\ q_k \end{Bmatrix}$$

Test/Analysis Correlation – Cont'd

Comparing generalized stiffness matrices for *MHMB* and *RFMB* in the component "test-space" *(q1-space):*

$$\kappa_1^{MHMB} = \begin{bmatrix} \kappa_{bb} & \kappa_{bc} & 0 \\ \kappa_{cb} & \kappa_{cc} & \kappa_{ck} \\ 0 & \kappa_{kc} & \lceil \omega_k^2 \rfloor \end{bmatrix} \qquad \kappa_1^{RFMB} = \begin{bmatrix} \kappa_{bb} & & \\ & g_{cc}^R & \\ & & \lceil \omega_k^2 \rfloor \end{bmatrix}$$

$$\kappa_{ck} = \kappa_{kc}{}' = \psi_c^{C\,'} k \, \phi_k^N$$

The RFMB generalized stiffness is block diagonal (no cross-coupling) and involves free-boundary residual flexibility which can be determined from component modal testing with time-tested methods. On the other hand, the MHMB (also HMB and RFMB-V) generalized stiffness matrices contain cross-coupling terms that can not be determined from component testing.

Rank-Deficiency Considerations in MB Methods

The rank of the mixed-boundary methods can be best studied by assuming an un-truncated normal mode set

- Since practical application of all component reduction methods often involve a significant level of normal mode truncation, this section is only of an academic value

For the *HMB* method, an un-truncated normal mode set will implicitly contain both the constraint modes as well as the inertia-relief modes resulting in up to $c + r$ singularities for the mixed-boundary and all-free-boundary cases

- For the all-fixed-boundary case, *HMB* can result up to r singularities for an un-truncated normal mode set

For *MHMB*, *RFMB*, and *RFMB-V* methods, an un-truncated normal mode set will result in up to c singularities for the mixed-boundary case

- Zero singularities for the all-fixed-boundary case
- Up to c singularities for the all-free-boundary case

Note: c denotes the number of dofs in the free-boundary set, r the inertia-relief set

Rank-Deficiency – Cont'd

Certain special cases can result in a rank-deficiency of a similar nature as that of employing an un-truncated normal mode set

- These special cases may very well fit into the category of <u>user misapplication</u> of mixed-boundary methods

Examples of user misapplication

- Mechanism dofs placed in the free-boundary set
  - These mechanism rigid-body modes will be implicitly contained in the constraint modes resulting in linear dependencies with the mechanism rigid-body normal modes in both *HMB* as well as *MHMB*
  - Similarly, but via a different mathematical route, these mechanism dofs result in rank-deficiencies in the *RFMB* and *RFMB-V* methods, since the mechanism rigid-body modes will contain the total flexibility for the dofs, leaving zero residual flexibility

Rank-Deficiency – Cont'd

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Examples of user misapplication

- Seismic masses placed in the free-boundary set (user intends to base-drive component; however, seismic masses are place on free-boundary dofs, forcing them to act like fixed-boundary dofs)
  - Seismic masses will force the free-boundary dofs to become "node points" in the normal modes, resulting in both zero total and zero residual flexibility for the subject dofs
  - From a methods equivalence argument, it is clear that the *HMB* and *MHMB* will have the same order singularities

Solution

- Place mechanism and seismic mass dofs on the fixed-boundary set to avoid singularities in mixed-boundary methods

Rank-Deficiency – Cont'd

Simple proof *RFMB* is not rank-deficient for a truncated set of modes [Theorems from reference 13]

- *THEOREM 9 – If B=PAQ, where P and Q are nonsingular matrices, then A and B are equivalent*
- *THEOREM 11 – Two (mxn) matrices are equivalent if and only if they have the same rank*

Rank-Deficiency – Cont'd

Referring to p. 14 of briefing, it is clear that *T1* has a rank of *b+c+k*, i.e., fully-ranked, since it is composed of linearly independent vector sets

Per *THEOREM 9*, the matrix multiplication *T1 x T2* will result in the *RFMB* transformation being of the same rank as (equivalent matrices having the same rank per *THEOREM 11*) if the *T2* matrix is nonsingular

- In other words, the *T2* matrix is substituted as the *Q* matrix in *THEOREM 9*, the *A* matrix being *T1* and the *P* matrix being identity. It is easily demonstrated that the *T2* matrix is nonsingular, i.e.,

$$T_2^{-1} = \begin{bmatrix} I & 0 & 0 \\ -g_{cc}^{R^{-1}}\psi_{cb}^{C} & -g_{cc}^{R^{-1}} & -g_{cc}^{R^{-1}}\phi_{ck}^{N} \\ 0 & 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 & 0 \\ -\psi_{cb}^{C} & -g_{cc}^{R} & -\phi_{ck}^{N} \\ 0 & 0 & I \end{bmatrix}$$

Q.E.D.

Example 1 - Benfield Truss Problem

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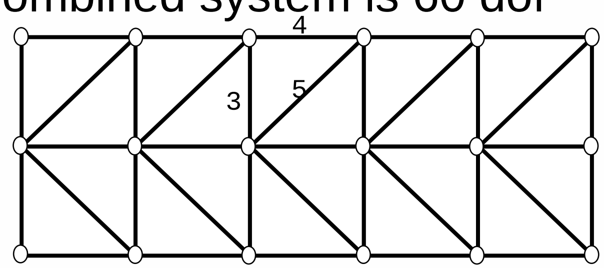
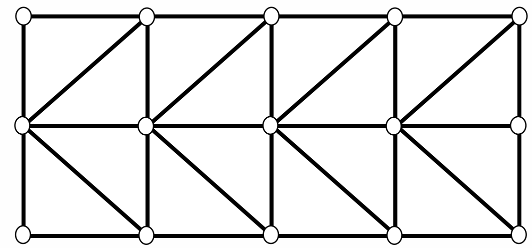
Classical problem to gauge the accuracy of CMS methods

- Each joint has 2 in-plane dof
- Interface is 6 dof (3 dof redundant)
- Combined system is 60 dof



5 EQUAL BAYS
SECTION A
36 dof



4 EQUAL BAYS
SECTION B
30 dof

Example 1 – Benfield Truss (Percent Frequency Error)

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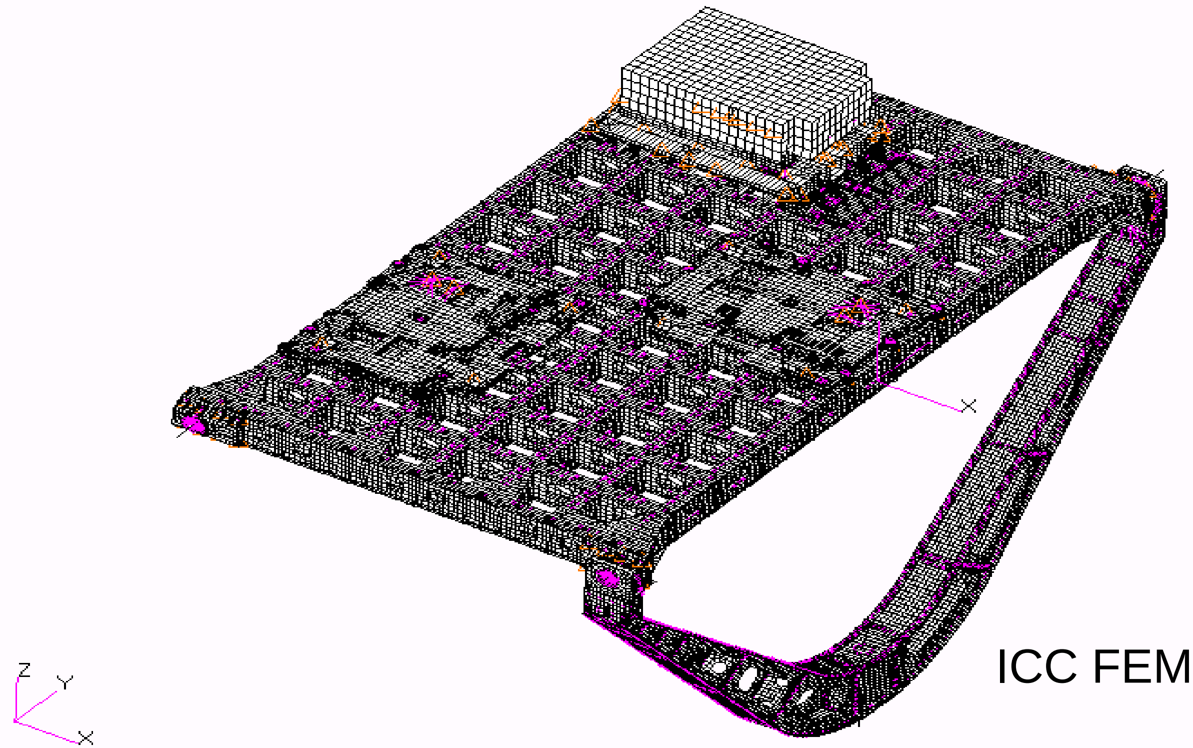
Elastic mode number	Craig-Bampton fixed-boundary normal modes *	Rubin free-boundary normal modes **	*MHMB* all-fixed-boundary normal modes *	*MHMB* all-free-boundary normal modes	*RFMB* all-fixed-boundary normal modes *	*RFMB* all-free-boundary normal modes **
1	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00
3	0.02	0.01	0.02	9.25	0.02	0.01
4	0.01	0.00	0.01	0.00	0.01	0.00
5	0.01	0.00	0.01	0.00	0.01	0.00
6	0.01	0.01	0.01	0.01	0.01	0.01
7	1.22	0.03	1.22	0.80	1.22	0.03
8	0.09	0.03	0.09	0.02	0.09	0.03
9	0.28	0.07	0.28	0.07	0.28	0.07
10	0.10	0.46	0.10	0.42	0.10	0.46
11	0.18	0.04	0.18	0.03	0.18	0.04
12	0.08	1.69	0.08	1.53	0.08	1.69
13	0.31	1.58	0.31	5.23	0.31	1.58
14	0.69	1.72	0.69	4.29	0.69	1.72
15	7.06	11.00	7.06	13.60	7.06	11.00
16	2.03	8.62	2.03	28.66	2.03	8.62
17	4.12	20.37	4.12	39.83	4.12	20.37
18	14.63	21.61	14.63	41.55	14.63	21.61
19	17.04	31.23	17.04	53.41	17.04	31.23
20	32.05	36.59	32.05	46.22	32.05	36.59
21	93.51	94.04	93.51	94.09	93.51	94.04

Section A – 11 normal modes
Section B – 7 normal modes

* Identical Results
** Identical Results

Example 2 – 650,000 dofs Integrated Cargo Carrier (ICC) Finite Element Model



ICC FEM supplied by EADS

Example 2 – *RFMB* vs. Craig-Bampton Percent Frequency Error

Mode #	Exact Frequency (Hz)	*RFMB*	Guyan	Guyan +8 Modes (100 Hz)	Guyan +15 Modes (150 Hz)	Guyan +24 Modes (200 Hz)	Guyan +42 Modes (250 Hz)	Guyan +74 Modes (300 Hz)	Guyan +242 Modes (400 Hz)
1	18.06	0.00	0.79	0.53	0.33	0.02	0.01	0.01	0.00
2	21.29	0.00	2.94	0.15	0.07	0.04	0.03	0.02	0.00
3	30.07	0.00	2.53	1.39	0.42	0.09	0.04	0.03	0.01
4	32.87	0.00	5.16	1.23	0.46	0.08	0.06	0.05	0.00
5	47.77	0.00	32.52	0.37	0.11	0.04	0.02	0.01	0.01
6	50.40	0.00	34.70	0.42	0.14	0.04	0.03	0.02	0.00
7	53.70	0.00	71.87	1.16	0.33	0.14	0.09	0.05	0.02
8	59.67	0.00	69.60	1.37	0.36	0.22	0.11	0.06	0.02
9	67.25	0.00	74.04	3.01	1.00	0.23	0.13	0.06	0.04
10	82.43	0.00	48.09	4.70	1.42	0.97	0.55	0.34	0.12
11	84.01	0.00	53.48	7.78	0.71	0.24	0.17	0.11	0.05
12	86.65	0.00	52.11	6.08	2.05	1.31	0.78	0.53	0.17
13	90.93	0.00	63.60	9.44	3.02	0.74	0.33	0.21	0.12
14	92.91	0.00		13.26	1.77	0.53	0.36	0.19	0.08
15	101.94	0.00		16.91	3.13	2.86	2.75	1.56	0.23
16	105.69	0.00		18.69	1.64	1.21	0.99	0.17	0.03
17	108.07	0.00		19.83	4.65	3.36	2.82	0.76	0.21
18	113.95	0.00		19.27	7.57	3.82	3.35	1.89	0.29
19	119.36	0.00			7.72	4.87	2.28	1.41	0.44
20	130.08	0.00			1.26	0.68	0.42	0.21	0.09
21	131.51	0.00			7.96	0.86	0.51	0.33	0.15
22	139.00	0.00				2.94	1.64	0.90	0.41
23	141.07	0.00				2.99	1.26	0.78	0.39
24	147.62	0.00						1.58	0.76

Note: The 24 mode *RFMB* model execution took just **59 minutes** compared to **4 hours and 56 minutes** it took to develop the 242 mode Craig-Bampton Model (not including the time for the required convergence testing of the Craig-Bampton). Both problems executed on same high-speed work station executing MSC.Nastran.

MB Methods Comparison - Summary

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Desired Characteristics for a Mixed-Boundary Method	HMB	MHMB	RFMB	RFMB-V
1. Accurate for All Component Boundary Conditions	x	x	✓	✓
2. Explicit Inclusion of Boundary Physical Coordinates	x	✓	✓	✓
3. Component Independence	✓	✓	✓	✓
4. Statically Complete	✓	✓	✓	✓
5. Test Compatibility	x	x	✓	x

HMB – Hintz Mixed-Boundary
MHMB – modified-Hintz Mixed-Boundary
RFMB – Residual Flexibility Mixed-Boundary
RFMB-V – Residual Flexibility Mixed-Boundary Variant

Concluding Remarks

It was demonstrated that the *RFMB* method is accurate for the entire range of component boundary dofs representation, from all-fixed- to mixed-, to all-free-boundaries

- *RFMB* exactly equal to Craig-Bampton for all-fixed- and Rubin for all-free-boundary cases

It was demonstrated that the *RFMB* method generalized stiffness matrix, which involves no cross-coupling terms, is the only candidate for direct determination/validation from component testing

It was demonstrated that the *RFMB* method is <u>not</u> rank-deficient for a truncated set of normal modes

- User misapplications of a method does not constitute rank-deficiency!

Concluding Remarks

RFMB method utilized by ASD to develop the ICC Dynamic Math Models and Output Transformation Matrices (EADS customer) for multiple design cycles and

- **STS-114 Verification Loads Analysis**

MSC.Nastran has implemented the *RFMB* method in its latest release of MSC.Nastran 2004 as *default*

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