# The motion of a single wheelset along a tangent track for single and double contact

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## 1. Introduction

In the present paper we discuss the motion of a single wheelset which is elastically connected to a vehicle body. The track along which the wheelset is moving, is supposed to be tangent and purely straight (without irregularities). The body moves with the constant speed $V$ in a direction parallel to the track.

It is interesting to mention that both the restriction to a tangent track and the restriction to a translational motion are not essential: it is quite possible to draw up a theory for the motion along a curved track of any shape of a wheelset connected to a vehicle body which presents parasitic (lateral, vertical and rolling) motions [1]. As we like to emphasize in the present paper the geometric contact theory, we here shall restrict ourselves to the model described in the previous paragraph.

Our starting-point is the division of the various geometric, kinematic and dynamic quantities into quantities of order zero ("O0") like the wheelbase and the wheel radius, quantities of order one ("O1") like the parasitic displacements and rotations of the wheelset and higher order quantities ("O2" etc.). Then in the equations of motion we restrict ourselves to O0 and O1 quantities and this procedure gives rise to considerable simplifications. However, our model remains to be strongly non-linear because the conicity is O0 rather than O1, and this results in a strong non-linearity of the geometric contact. Besides, the physical contact, too, is strongly non-linear.

Further on, we shall restrict ourselves to the case that the track and the wheelset profiles are symmetric.

Note also that in the paper overlining indicates a geometric vector, whereas underlining indicates an algebraic vector.

## 2. Outline of the model

In fig. 2-1 we have schematically indicated the positions of the vehicle body and the wheelset. In any central position of the wheelset its centre is on the straight line $\omega\xi$ which is parallel to the track. In the marked position, in which $\omega\xi = s^*$, the centre of the wheelset is the point $O$ when it is in its central position, and in the point $O^*$, situated in the plane $(y,O,z)$, when it is in any other position. In sec. 3 we shall discuss more in detail where the centre of the system $(O^*,x^*,y^*,z^*)$ is situated.

The coordinate system $(O_b,x_b,y_b,z_b)$ is connected with the body, $O_b$ being situated on the axis $\omega\xi$. This point is connected to the wheelset centre by a linear spring. The system $(O_b,x_b,y_b,z_b)$ purely translates along the track.

The effective rigidities of the longitudinal, lateral and vertical springs are $c_x, c_y$ and $c_z$ respectively. In railway practice the spring forces apply at the axlebox centres $B_1$ and $B_2$, shown in fig. 2-2, and for each of these points the rigidities are $\frac{1}{2}c_x, \frac{1}{2}c_y$ and $\frac{1}{2}c_z$. From this we easily can derive the forces and moments shown in fig. 2-2. Note that in the central position the body exerts the gravity force G at the wheelset; in any other position this force amounts to the value $G - c_z w$ .

## 3. The geometric contact

The wheelset displacement is shown in fig. 3-1. The translation with respect to $(O,x,y,z)$ is described by $v$ and $w$ and the rotation is partially described by the angles $\varphi$ and $\psi$, to which the rotation $(-\theta)$ about the axis $O^*y^*$ should be added. The latter rotation does not influence the geometric contact and we can restrict ourselves to the O1 quantities $v$, $w$, $\varphi$ and $\psi$, which describe the wheelset motion together with the OO quantities $s^*$ and $\theta$.

Clearly, there must exist two constraint relations between $v$, $w$, $\varphi$ and $\psi$. Henceforth, we shall consider $v$ and $\psi$ as the <u>independent</u> and $w$ and $\varphi$ as the <u>dependent coordinates</u>, so that we can write

$$w = w(v,\ \psi), \quad \varphi = \varphi(v,\ \psi) . \tag{3-1}$$

The complete model is described with seven coordinates $s^*, v, w, \varphi, \theta, \psi$ and the body displacement $s$. But as the body velocity is equal to $V$($V$ being constant), we can write

$$s = Vt . \tag{3-2}$$

This enables us to replace the OO coordinate $s^*$ with the O1 coordinate

$$u = s^* - Vt . \tag{3-3}$$

In the same way we also can replace the OO coordinate $\theta$ with the O1 coordinate

$$\chi = s/r - \theta , \tag{3-4a}$$

so that also

$$\chi = Vt/r - \theta . \tag{3-4b}$$

In this way the wheelset motion can be described by <u>six</u> O1 coordinates: $u, v, w, \varphi, \chi$ and $\psi$.

The relation (3-2) is a constraint equation as well. We shall always take it into account. Thus our system has <u>four</u> degrees of freedom when the constraint equations (3-1) are taken into account and <u>six</u> degrees of freedom when they are not taken into account.

We now concentrate on the latter constraint equations. We call $A_j$ the contact point on the rail surface and $A_j^*$ the contact point on the wheel surface; note that always $j=1$ for the right-hand side and $j=2$ for the left-hand side of the system. Then we can find <u>six</u> equations from the condition that $A_j$ and $A_j^*$ coincide and <u>four</u> equations from the condition that in the contact point $A_j$ the normal $\overline{n}_j$ on the rail surface has the same direction as that of the normal $\overline{n}_j^*$ on the wheel surface in $A_j^*$.

For the description of the rail surface we introduce the coordinate systems $(A_{0j}, \xi_j, \eta_j, \zeta_j)$ shown in fig. 3-2. Clearly,

$$x_j = \xi_j, \quad y_j = \pm(b - \eta_j), \quad z_j = r + \zeta_j \quad (j = 1,2) . \tag{3-5}$$

We now can describe the profile by the function $f$:

$$\zeta_j = f(\eta_j) \quad (j = 1,2) . \tag{3-6}$$

Then the conicity $\gamma_j$ is determined by

$$\tan\gamma_j = \left(\frac{d\zeta_j}{d\eta}\right)_{\eta=\eta_j} = \left(\frac{df}{d\eta}\right)_{\eta=\eta_j} = f'(\eta_j), (j = 1,2) \tag{3-7}$$

For the rail surface j there exists the relation

$$F_j = 0 , \tag{3-8}$$

between $\xi_j, \eta_j, \zeta_j$, where

$$F_j = \zeta_j - f(\eta_j) . \tag{3-9}$$

Here $F_j$ should also contain the terms O2 because for determining the normal direction we have to differentiate $F_j$. Fortunately, $F_j$(3-8) contains the terms of any order.

Now we can write for the directions of $\overline{n}_j$ :

$$\underline{n}_j = \left[n_{xj}, n_{yj}, n_{zj}\right]^T , \tag{3-10}$$

where

$$n_{xj} = \frac{\partial F_j / \partial x_j}{\sqrt{\left(\partial F_j/\partial x_j\right)^2 + \left(\partial F_j/\partial y_j\right)^2 + \left(\partial F_j/\partial z_j\right)^2}}, \quad n_{yj} = \frac{\partial F_j / \partial y_j}{\sqrt{\left(\partial F_j/\partial x_j\right)^2 + \left(\partial F_j/\partial y_j\right)^2 + \left(\partial F_j/\partial z_j\right)^2}}$$

$$n_{zj} = \frac{\partial F_j / \partial z_j}{\sqrt{\left(\partial F_j/\partial x_j\right)^2 + \left(\partial F_j/\partial y_j\right)^2 + \left(\partial F_j/\partial z_j\right)^2}} \tag{3-11}$$

Together with (3-9), (3-5) and (3-7) we obtain

$$\frac{\partial F_j}{\partial x_j} = \frac{\partial F_j}{\partial \xi_j} = 0, \quad \frac{\partial F_j}{\partial y_j} = \mp\frac{\partial F_j}{\partial \eta_j} = \pm\tan\gamma_j + O_2, \quad \frac{\partial F_j}{\partial z_j} = \frac{\partial F_j}{d\zeta_j} = 1 + O_2 , \tag{3-12}$$

$$\sqrt{\left(\partial F_j / \partial x_j\right)^2 + \left(\partial F_j / \partial y_j\right)^2 + \left(\partial F_j / \partial z_j\right)^2} = \cos^{-1}\gamma_j + O_2 , \tag{3-13}$$

$$n_{xj} = 0, \quad n_{yj} = \pm \sin\gamma_j + O_2, \quad n_{zj} = \cos\gamma_j + O_2 . \tag{3-14}$$

For the wheelset the surface function is more complicated: see fig. 3-3. In the right-hand part we have indicated the contact point $A_1^*$. We have introduced the additional axis $O^*z_1^{**}$ and in the left-hand part we have shown the plane $(y^*, O^*, z_1^{**})$. We call $r_1^*$ the distance from $A_1^*$ to $O^*y^*$. Now the profile curve can be indicated by means of the system $(\eta, A_{01}^*, \rho_1^*)$. For both values of $j$ we now can write down the counterparts of (3-5)-(3-7):

$$x_j^* = \xi_j^*, \quad y_j^* = \pm(b - \eta_j^*), \quad z_j^* = r + \zeta_j^*, \quad r_j^* = r + \rho_j^* \quad (j = 1,2) , \tag{3-15}$$

$$\rho_j^* = f(\eta_j^*) \quad (j=1,2)\ , \tag{3-16}$$

$$\tan\gamma_j^* = \left(\frac{d\rho_j^*}{d\eta^*}\right)_{\eta^*=\eta_j^*} = \left(\frac{df^*}{d\eta^*}\right)_{\eta^*=\eta_j^*} = f^{*'}(\eta_j^*) \quad (j=1,2) \tag{3-17}$$

Moreover, we have

$$r_j^{*2} = x_j^{*2} + z_j^{*2} \tag{3-18}$$

From (3-15) and (3-18) we can find the wheel surface equations

$$F_j^* = 0 \tag{3-19}$$

with

$$F_j^* = \tfrac{1}{2}\left\{\xi_j^* + (r+\zeta_j^*)^2 - (r+\rho_j^*)^2\right\} + O_3 \tag{3-20}$$

Writing this out yields

$$r(\zeta_j^* - \rho_j^*) = O_2\ , \tag{3-21}$$

which means that

$$\rho_j^* = \zeta_j^* + O_2 \tag{3-22}$$

This means that in all relations, with the exception of (3-20), we now can replace $\rho_j^*$ with $\zeta_j^*$. So we can replace (3-16) and (3-17) with

$$\zeta_j^* = f^*(\eta_j^*) \quad (j=1,2)\ , \tag{3-23}$$

$$\tan\gamma_j^* = \left(\frac{d\zeta_j^*}{d\eta^*}\right)_{\eta^*=\eta_j^*} = \left(\frac{df^*}{d\eta^*}\right)_{\eta^*=\eta_j^*} = f^{*'}(\eta_j^*), (j=1,2) \tag{3-24}$$

and we find by means of (3-20), (3-15) and (3-17):

$$\frac{\partial F_j^*}{\partial x_j^*} = \frac{\partial F_j^*}{\partial \xi_j^*} = \xi_j^*,\quad \frac{\partial F_j^*}{\partial y_j^*} = \mp\frac{\partial F_j^*}{\partial \eta_j^*} = \pm(r+\zeta_j^*)\tan\gamma_j^*,\quad \frac{\partial F_j}{\partial z_j^*} = \frac{\partial F_j}{d\zeta_j^*} = r+\zeta_j\ , \tag{3-25}$$

$$\sqrt{\left(\partial F_j^*/\partial x_j^*\right)^2 + \left(\partial F_j^*/\partial y_j^*\right)^2 + \left(\partial F_j^*/\partial z_j^*\right)^2} = (r+\zeta_j^*)\cos^{-1}\gamma_j^* \tag{3-26}$$

and

$$n_{xj}^* = \frac{\xi_j^*\cos\gamma_j^*}{r},\quad n_{yj}^* = \pm\sin\gamma_j^*,\quad n_{zj}^* = \cos\gamma_j^*, \tag{3-27}$$

We still have to determine the relation between $(O, x, y, z)$ and $(O^*, x^*, y^*, z^*)$ From fig. 3-1 we derive

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ v \\ w \end{bmatrix} + \underline{G}\begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix}, \tag{3-28}$$

where the matrix $\underline{G}$ (to be distinguished from the weight force $G$) is equal to

$$\underline{G} = \begin{bmatrix} \cos\psi & -\cos\varphi\sin\psi & \sin\varphi\sin\psi \\ \sin\psi & \cos\varphi\cos\psi & -\sin\varphi\cos\psi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} 1 & -\psi & 0 \\ \psi & 1 & -\varphi \\ 0 & \varphi & 1 \end{bmatrix} + O_2\ . \tag{3-29}$$

Together with (3-5) and (3-15) the condition $A_j = A_j^*$ yields with terms of O0 and O1:

$$\xi_j = \xi_j^* \mp b\psi, \tag{3-30a}$$

$$\pm b \mp \eta_j = v \pm b \mp \eta_j^* - r\varphi \ , \tag{3-30b}$$

$$r + \zeta_j = w + l\varphi + r + \zeta_j^* \ . \tag{3-30c}$$

The condition $\vec{n}_j = \vec{n}_j^*$ reads in algebraic notation:

$$\underline{n}_j^* = \underline{G}^T \underline{n}_j \tag{3-31}$$

or together with (3-29), (3-14) and (3-27) in terms of O0 and O1 (like everywhere in the sequel):

$$\frac{\xi_j^* \cos\gamma_j^*}{r} = \mp\psi \sin\gamma_j \ , \tag{3-32a}$$

$$\pm\sin\gamma_j^* = \pm\sin\gamma_j + \varphi\cos\gamma_j \ , \tag{3-32b}$$

$$\cos\gamma_j^* = \pm\varphi\sin\gamma_j + \cos\gamma_j \ , \tag{3-32c}$$

the two latter equations being identical: they can be replaced with

$$\gamma_j^* = \gamma_j \pm \varphi \ . \tag{3-33}$$

Next, (3-31a-c), (3-32a) and (3-33) can be rewritten like

$$r\varphi \mp (\eta_j - \eta_j^*) = v \ , \tag{3-34a}$$

$$w \pm b\varphi - (\zeta_j - \zeta_j^*) = 0 \ , \tag{3-34b}$$

$$\pm\varphi + \gamma_j - \gamma_j^* = 0 \ , \tag{3-34c}$$

$$\xi_j = \mp b_j \varphi, \quad \xi_j^* = \pm r\psi \tan\gamma_j \ , \tag{3-35a-b}$$

with

$$b_j = b - r\tan\gamma_j . \tag{3-36}$$

In (3-34a-c) $\zeta_j, \zeta_j^*, \gamma_j$ and $\gamma_j^*$ only depend on $\eta_j$ and $\eta_j^*$, to wit through (3-6), (3-23), (3-7) and (3-24). Thus these six equations contain no more than six unknowns: $w, \varphi, \eta_1, \eta_2, \eta_1^*$ and $\eta_2^*, v$ being given.

The equations are strongly non-linear; they can be solved approximately by means of the Newton-Raphson procedure. For, their Jacobian can be determined in an analytic way: the relations (3-7) and (3-24) can be rewritten like

$$\gamma_j = \arctan(\zeta_j'), \quad \gamma_j^* = \arctan(\zeta_j^{*\prime}), \tag{3-37}$$

so that

$$r_{yj} = (\zeta_j'' \cos^3\gamma_j)^{-1}, \quad r_{yj}^* = (\zeta_j^{*\prime\prime} \cos^3\gamma_j^*)^{-1} \ , \tag{3-38}$$

and

$$\gamma_j' = (r_{yj}\cos\gamma_j)^{-1}, \quad \gamma_j^* = (r_{yj}^*\cos\gamma_j^*)^{-1} \ , \tag{3-39}$$

$r_{yj}$ and $r_{yj}^*$ respectively being the rail profile and wheel profile radii of curvature.

In the further calculations we also will need the expressions for $w,_v = \partial w/\partial v$ and $\varphi,_v = \partial\varphi/\partial v$ (note that in our first order theory $w,_\psi = \partial w/\partial\psi = 0$ and $\varphi,_\psi = \partial\varphi/\partial\psi = 0$). Differentiating (3-34a-b) with respect to $v$ yields

$$w_{,v} = \frac{b(\tan\gamma_1 - \tan\gamma_2)}{b_1 + b_2} + O_1, \quad \varphi_{,v} = -\frac{\tan\gamma_1 + \tan\gamma_2}{b_1 + b_2} + O_1 . \tag{3-40}$$

The Newton-Raphson procedure boils down to the following algorithm. We always can write the non-linear equations to be solved like

$$\underline{f}(x, \underline{y}) = \underline{0} , \tag{3-41a}$$

$x$ being a scalar and $\underline{f}$ and $\underline{y}$ column vectors with $n$ elements (so that in our case

$$n = 6, \quad x = v, \quad \underline{y} = \left[w, \varphi, \eta_1, \eta_2, \eta_1^*, \eta_2^*\right]^T . \tag{3-41b}$$

Now assume that $\underline{y}_0$ is an approximate solution of (3-40) for $x = x_0$. Then an improved solution $\underline{y}_1 = \underline{y}_0 + \Delta\underline{y}$ can be found by substituting these $x_0$ and $\underline{y}$ into (3-40), which leads to

$$\underline{f}\left(x_0, \underline{y}_0\right) + \frac{\partial \underline{f}}{\partial \underline{y}} \Delta\underline{y} = \underline{0}$$

when we neglect higher order terms. Thus the improved solution reads

$$\underline{y}_1 = \underline{y}_0 - \left(\frac{\partial \underline{f}}{\partial \underline{y}}\right)^{-1} \underline{f}\left(x_0, y_0\right) \tag{3-42}$$

and this procedure can be repeated at will.

When the rail-wheel contact only can take place in one single point, we can always start with the value $v = 0$, increase $v$ gradually and calculate the other geometric variables for each value of $v$. (Note that $w$ is an even and $\varphi$ is an odd function of $v$.) However, this method gives rise to difficulties when the combination of profiles also admits double-point contact: see fig. 3-4, in which we have shown the relation between $v$ and $\varphi$ for such a case for positive values of $v$. Then there are two turning-points: $P_1$ and $P_2$, whereas in the point $P_{12}$ there is a double contact: in that point $\eta_j, \eta_j^*, \zeta_j, \zeta_j^*, \gamma_j, \gamma_j^*, r_{yj}$ and $r_{yj}^*$ have two different values.

In that case $v$ decreases rather than increases between the points $P_1$ and $P_2$, when the contact points moves continuously along the rail and the wheel profiles. However, on the sections $P_{12}P_1, P_1P_2$ and $P_2P_{12}$ the wheel penetrates into the rail, which physically is impossible. This means that these sections have no physical meaning and that they have to be left out of consideration. In $P_{12}$ the contact point jumps from one position to another one.

On the other hand the three afore-mentioned sections enable us to determine the double point in an elegant way. For this purpose we introduce an additional variable: the arclength $s$. Clearly,

$$x = x(s), \quad \underline{y} = \underline{y}(s) . \tag{3-43}$$

Differentiating (3-40) with respect to $s$ yields

$$d\underline{f} = \frac{\partial \underline{f}}{\partial x} dx + \frac{\partial \underline{f}}{\partial \underline{y}} d\underline{y} = \underline{0} \ ,$$

viz.

$$\frac{\partial \underline{f}}{\partial x} x' + \frac{\partial \underline{f}}{\partial \underline{y}} \underline{y}' = \underline{0} \ , \qquad (3\text{-}44)$$

the prime indicating a differentiation with respect to $s$. Moreover, because

$$(ds)^2 = (dx)^2 + d\underline{y}^T d\underline{y} \ , \qquad (3\text{-}45)$$

we have in addition

$$(x')^2 + \underline{y}'^T d\underline{y}' = 1 \ . \qquad (3\text{-}46)$$

For the $n$+1 equations (3-44) and (3-46) we can solve $x'$ and $\underline{y}'$, and this can be done in an easy way because the $n$ equations (3-44) are linear in $x'$ and the components of $\underline{y}'$.

We put

$$\frac{d\underline{y}}{dx} = \underline{c} \ , \qquad (3\text{-}47)$$

then

$$\underline{y}' = \underline{c} x' \ . \qquad (3\text{-}48)$$

Together with (3-44) we obtain

$$\left( \frac{\partial \underline{f}}{\partial x} + \frac{\partial \underline{f}}{\partial \underline{y}} \underline{c} \right) x' = 0 \ , \qquad (3\text{-}49)$$

so that, provided that $x' \neq 0, \underline{c}$ is determined from

$$\underline{c} = -\left( \frac{\partial \underline{f}}{\partial \underline{y}} \right)^{-1} \frac{\partial \underline{f}}{\partial x} \ . \qquad (3\text{-}50)$$

Combining this with (3-46) yields an equation for $x'$ from which $x'$ can be solved:

$$x' = \pm \frac{1}{\sqrt{1 + \underline{c}^T \underline{c}}} \ . \qquad (3\text{-}51)$$

The above-mentioned method has been indicated by R. Seydel[2 p. 111-113]. However, he does not indicate how in (3-51) the sign has to be chosen. But this can be done quite easily by comparing the direction of the vector $\left[x', \underline{y}'^T\right]^T$ in the $(n+1)$-dimensional space with the direction of the vector $\left[x'_{old}, \underline{y}'_{old}{}^T\right]^T$ belonging to the previous value of $s$ and this leads to the condition

$$\left(1 + \underline{c}_{old}{}^T \underline{c}\right) x'_{old} x' > 0 \ , \qquad (3\text{-}52)$$

from which we easily can derive the sign (+ or -) in (3-51).

Numerically, the vector $\underline{c}$ determines the tangent to the curve in the $(n+1)$-dimensional space: the predictor part of the procedure. When this vector is found, we increase the value of $s$ and choose a point on this tangent; this point belongs to an approximate solution of (3-40), which can be used for starting the NR-procedure. When that procedure diverges, we have to decrease the value of $s$, to choose a point more in the neighbourhood of the previous solution and to try again, etc. This is the corrector part of the program.

Once the points $P_1$ and $P_2$ in fig. 3-4 have been found, $P_2$ can be used for finding the coordinates of the double-point $P_{12}$. For this purpose we use the NR-procedure on the section $OP_1$ and on the section starting in $P_2$ on which $v'>0$. Then we determine the values of $\varphi$ at issue and their difference: when the latter vanishes, we are in the neighbourhood of $P_{12}$. The exact position of $P_{12}$ best can be found by writing down the equations (3-34a-c), which now determine the double point: there now are nine of such equations for the nine unknowns $v,w,\varphi,\eta_1,\eta_2,\eta_3,\eta_1^*,\eta_2^*,\eta_3^*$; here the index 3 refers to the second contact point at the right-hand side, when we suppose $v$ to be positive.

## 4. Dynamics

We now shall derive the equations of motion for the system. This can be done by means of the well-known Newton-Euler equations, written down for an inertial reference frame:

$$\dot{\underline{p}}=\underline{f}, \quad \dot{\underline{l}}=\underline{m} . \tag{4-1}$$

Here $\underline{p}$ is the wheelset momentum, $\underline{l}$ its moment of momentum, $\underline{f}$ the resultant of the forces applied at the wheelset and $\underline{m}$ the moment of this force about the mass centre of the wheelset. The equations (4-1) are valid for the vehicle body as well.

The coordinate system $(O_b,x_b,y_b,z_b)$ of fig. 2-1 translates in the $x$-direction with the constant speed $\dot{s}=V$ and can be used in (4-1). The radius vector $\underline{r}_0$ and the velocity of $O^*$ with respect to $(O_b,x_b,y_b,z_b)$ are equal to

$$\underline{r}_0=\left[u,v,w\right]^T , \tag{4-2}$$

$$\underline{v}_0=\left[\dot{u},\dot{v},\dot{w}\right]^T . \tag{4-3}$$

Thus we obtain

$$\underline{p}=m\underline{v}_0 , \quad \underline{b}=\widetilde{\underline{r}}_0\underline{p}+\underline{b}_0 , \tag{4-4}$$

$m$ being the wheelset mass and $\underline{b}_0$ the moment of momentum with respect to a coordinate system with the origin $O^*$ and axes parallel to $(O_b,x_b,y_b,z_b)$. From (4-2)-(4-4) we deduce that

$$\widetilde{\underline{r}}_0\underline{p}=O_2 , \tag{4-5}$$

so that this term can be omitted when we restrict ourselves to 00 and 01 terms. In the same approximation we find

$$\underline{b}_0 = \underline{G}\underline{I}^* \underline{\omega}_w^* \tag{4-6}$$

with $\underline{G}$ (3-29); see also fig. 3-1. Here the inertia tensor $\underline{I}^*$ is equal to

$$\underline{I}^* = \begin{bmatrix} I & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{4-7}$$

$I$ being the wheelset moment of inertia with respect to $O^*x^*$ and $O^*z^*$ and $I_y$ its moment of inertia with respect to $O^*y^*$; $w_w^*$ is the angular velocity of the wheelset with respect to $\left(O^*,x^*,y^*,z^*\right)$ and amounts to

$$\underline{\omega}_w^* = \left[\dot{\varphi},-\dot{\theta},\dot{\psi}\right]^T ,$$

so that because of (3-4b) also

$$\underline{\omega}_w^* = \left[\dot{\varphi},-V/r+\dot{\chi},\dot{\psi}\right]^T , \tag{4-8}$$

Altogether, (4-3)-(4-8) and (3-29) give rise to

$$\underline{p} = m\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix}, \quad \underline{b} = \underline{I}^*\begin{bmatrix} \dot{\varphi} \\ \dot{\chi} \\ \dot{\psi} \end{bmatrix} + \frac{I_y V}{r}\begin{bmatrix} \psi \\ -1 \\ -\varphi \end{bmatrix} . \tag{4-9}$$

Thus the equations of motion (4-1) read

$$m\begin{bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{bmatrix} = \underline{f}, \quad \underline{I}^*\begin{bmatrix} \ddot{\varphi} \\ \ddot{\chi} \\ \ddot{\psi} \end{bmatrix} + \frac{I_y V}{r}\begin{bmatrix} \dot{\psi} \\ 0 \\ -\dot{\varphi} \end{bmatrix} = \underline{m} . \tag{4-10}$$

In the sequel we shall perform all derivations with respect to the distance $s$, indicated by an apostrophe ('). Moreover, we combine the vectors $\underline{f}$ and $\underline{m}$ to the single force vector

$$\underline{k} = \begin{bmatrix} \underline{f} \\ \underline{m} \end{bmatrix} . \tag{4-11}$$

We also introduce the centrifugal force

$$\underline{k}_g = -\frac{I_y V^2}{r}\left[0,0,0,\psi',0,-\varphi'\right]^T . \tag{4-12}$$

Then the equations of motion (4-10) can be written like

$$\underline{M}V^2 \underline{q}'' = \underline{k}_g + \underline{k} , \tag{4-13}$$

where

$$\underline{M} = \begin{bmatrix} m\underline{E} & \underline{0} \\ \underline{0} & \underline{I}^* \end{bmatrix}, \tag{4-14}$$

$\underline{0}$ being the zero matrix and $\underline{E}$ the unity matrix, and

$$\underline{q} = [u, v, w, \varphi, \chi, \psi]^T . \tag{4-15}$$

The equations of motion (4-13) contain all the six coordinates. In the following way it is possible to find a set of equations of motion which only contain the independent coordinates. The vector of the latter coordinates is

$$\underline{q}^* = [u, \chi, v, \psi]^T . \tag{4-16}$$

By means of the constraint equations (3-1) we can find the relation

$$\underline{q}' = \underline{J}^* \underline{q}^{*'} \tag{4-17}$$

between the derivatives $\underline{q}'$ and $\underline{q}^*$; here $\underline{J}^*$ is the Jacobian matrix

$$\underline{J}^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & w_{,v} & \varphi_{,v} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T . \tag{4-18}$$

Now we have

$$\underline{q}'' = \underline{J}^* \underline{q}^{*''} + O_2 , \tag{4-19}$$

so that premultiplying (4-13) by $\underline{J}^{*T}$ results in

$$\underline{M}^* V^2 \underline{q}^{*''} = \underline{J}^{*T}(\underline{k}_g + \underline{k}) , \tag{4-20}$$

where

$$\underline{M}^* = \underline{J}^{*T}\, \underline{M}\underline{J}^* = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & I_y & 0 & 0 \\ 0 & 0 & m(1+w_{,v}^2) + I\varphi_{,v}^2 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}; \tag{4-21}$$

see (4-18) and (4-14).

Next we discuss the force $\underline{k}$: this is the sum of the impressed force $\underline{k}_m$ and the constraint force $\underline{k}_c$. Further on, the impressed forces can be divided into the spring force $\underline{k}_g$, the weight force $\underline{k}_w$ and the tangential force $\underline{k}_t$:

$$\underline{k} = \underline{k}_m + \underline{k}_c, \quad \underline{k}_m = \underline{k}_s + \underline{k}_w + \underline{k}_t . \tag{4-22}$$

The forces $\underline{k}_s$ and $\underline{k}_w$ easily are found by means of fig. 2-2:

$$\underline{k}_s = -\left[c_x u, c_y v, c_z w, c_z l_l^2 \varphi, 0, c_x l_l^2 \psi\right]^T , \tag{4-23}$$

$$\underline{k}_w = \left[0, 0, G, 0, 0, 0\right]^T . \tag{4-24}$$

The forces $\underline{k}_t$ and $\underline{k}_c$ are determined jointly. Observing fig. 4-1 we find

$$\begin{bmatrix} X_j \\ Y_j \\ Z_j \end{bmatrix} = -\underline{\Gamma}_j \begin{bmatrix} T_{xj} \\ T_{yj} \\ N_j \end{bmatrix} = -\underline{\Gamma}_{tj} \begin{bmatrix} T_{xj} \\ T_{yj} \end{bmatrix} - \underline{\Gamma}_{nj} N_j \quad (j = 1,2) \ , \tag{4-25}$$

where

$$\underline{\Gamma}_j = \left[\underline{\Gamma}_{tj}, \underline{\Gamma}_{nj}\right], \quad \underline{\Gamma}_{tj} = \begin{bmatrix} 1 & 0 \\ 0 & \cos\gamma_j \\ 0 & \mp\sin\gamma_j \end{bmatrix}, \quad \underline{\Gamma}_{nj} = \begin{bmatrix} 0 \\ \pm\sin\gamma_j \\ \cos\gamma_j \end{bmatrix} . \tag{4-26}$$

We call $\underline{r}_j^0$ the radius vector in $(O,x,y,z)$ from $O^*$ to the contact point $A_j$:

$$\underline{r}_j^0 = \left[\xi_j, \ \pm(b-\eta_j) - v, r + \zeta_j - w\right]^T + O_2 = \left[0, \ \pm b, \ r\right]^T + O_1 . \tag{4-27}$$

Then we can write

$$\underline{k}_t + \underline{k}_c = \sum_{j=1}^{2} \begin{bmatrix} \underline{E} \\ \underline{\tilde{r}}_j^0 \end{bmatrix} \begin{bmatrix} X_j \\ Y_j \\ Z_j \end{bmatrix} . \tag{4-28}$$

Combining this with (4-25) yields

$$\underline{k}_t = -\underline{D}_t \left[T_{x1}, T_{y1}, T_{x2}, T_{y2}\right]^T, \quad \underline{k}_c = -\underline{D}_n \left[N_1, N_2\right]^T \ , \tag{4-29a-b}$$

where

$$\underline{D}_t = \begin{bmatrix} \underline{\Gamma}_{t1} & \underline{\Gamma}_{t2} \\ \underline{\tilde{r}}_1^0 \underline{\Gamma}_{t1} & \underline{\tilde{r}}_2^0 \underline{\Gamma}_{t2} \end{bmatrix}, \quad \underline{D}_n = \begin{bmatrix} \underline{\Gamma}_{n1} & \underline{\Gamma}_{n2} \\ \underline{\tilde{r}}_1^0 \underline{\Gamma}_{n1} & \underline{\tilde{r}}_2^0 \underline{\Gamma}_{n2} \end{bmatrix} . \tag{4-30a-b}$$

The tangential forces $T_{x1}, \ldots, T_{y2}$ will be discussed in sec. 6, whereas the normal forces $N_1, N_2$ are constraint forces, which can be found in the following way. Inspecting (4-18), (4-30b), (4-26) and (4-27) yields the relation

$$\underline{J}^{*T} \underline{D}_n = \underline{0} \ , \tag{4-31}$$

which also can be derived from the fact that in a contact point the power of the constraint force is zero. Combining (4-29b) and (4-31) yields

$$\underline{J}^{*T} \underline{k}_c = \underline{0} \ . \tag{4-32}$$

From (4-19) and (4-22) we can reduce (4-13) to

$$\underline{MV^2} \underline{J}^* \underline{q}^{*\prime\prime} = \underline{k}_g + \underline{k}_s + \underline{k}_w + \underline{k}_t + \underline{k}_c . \tag{4-33}$$

Premultiplying by $\underline{D}_n^T \underline{M}^{-1}$ and using (4-31) yields the relation

$$\underline{D}_n^T \underline{M}^{-1}\left(\underline{k}_g + \underline{k}_s + \underline{k}_w + \underline{k}_t + \underline{k}_c\right) = \underline{0} \tag{4-34}$$

so that, together with (4-30b), $N_1$ and $N_2$ can be found from the equation

$$\underline{D}_n^T \underline{M}^{-1} \underline{D}_n \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \underline{D}_n^T \underline{M}^{-1}\left(\underline{k}_g + \underline{k}_s + \underline{k}_w + \underline{k}_t\right) . \tag{4-35}$$

The tangential forces depend on the creep and spin quantities, which can be found by considering the kinematics of the system, to be dealt with in sec. 5.

We still have to consider the equations of motion for the vehicle body. Because it is purely translating with the constant speed $V$, we are left with just one equation:

$$L = -c_x u \ , \tag{4-36}$$

$L$ being the tractive effort which has to be applied at the body for keeping its velocity constant.

## 5. Kinematics

The absolute velocity $\underline{v}_0$ of the wheelset mass centre $O^*$ is equal to the velocity (4-3) increased with the velocity of the system $(O_b, x_b, y_b, z_b)$:

$$\underline{v}_0 = V[1,0,0]^T + [\dot{u}, \dot{v}, \dot{w}]^T \ , \tag{5-1}$$

The wheelset angular velocity $\underline{\omega}$ is found from $\underline{\omega}_w^*$ (4-8):

$$\underline{\omega} = \underline{G}\underline{\omega}_w^* = \frac{V}{r}[\psi, -1, -\varphi]^T + [\dot{\varphi}, \dot{\chi}, \dot{\psi}]^T . \tag{5-2}$$

In the following way these results can be combined:

$$\begin{bmatrix} \underline{v}_o \\ \underline{\omega} \end{bmatrix} = \frac{V}{r}[r, 0, 0, \psi, -1, -\varphi]^T + \underline{J}^* V \underline{q}^{*\prime} \ ; \tag{5-3}$$

see (4-17).

In the contact point $j$ the velocity $\underline{v}_j$ amounts to

$$\underline{v}_j = \underline{v}_0 + \underline{\tilde{\omega}}\underline{r}_j^0 = \left[\underline{E}, -\underline{\tilde{r}}_j^0\right]\begin{bmatrix} \underline{v}_0 \\ \underline{\omega} \end{bmatrix} ; \tag{5-4}$$

evaluation by means of (4-27), (5-3) and (3-34b) yields

$$\underline{v}_j = V\begin{bmatrix} -\zeta_j^*/r & +u' + r\chi' \mp b\psi' \\ -\psi & +v' - r\varphi' \\ \pm\psi\tan\gamma_j & \mp(v' - r\varphi')\tan\gamma_j \end{bmatrix} . \tag{5-5}$$

Considering fig. 4-1 and making use of (3-40) we find the relation

$$\begin{bmatrix} W_{txj} \\ W_{tyj} \\ W_{nj} \end{bmatrix} = \underline{\Gamma}_j^T \underline{v}_j = V\begin{bmatrix} -\zeta_j^*/r \\ -\psi\cos^{-1}\gamma_j \\ 0 \end{bmatrix} + V\begin{bmatrix} 1r & 0 & \mp b \\ 00\frac{2b\cos^{-1}\gamma_j}{b_1+b_2} & 0 \\ 00 & 0 & 0 \end{bmatrix}\underline{q}^{*'} \quad . \tag{5-6}$$

so that $W_{nj}=0$, as it should be.

We also need the angular velocity $\omega_{nj}$ around $\bar{n}_j$; we easily find that

$$\omega_{nj} = \mp r^{-1}\sin\gamma_j + O_1 \; . \tag{5-7}$$

The O1 terms can be determined without difficulties; however, they can be omitted as compared with the O0 term.

## 6. The physical contact

In the present section we shall omit the contact point index $j$.

Kalker[3,4] has shown how in a contact point the tangential force can be found. At first it is necessary to determine the Hertz contact ellipse. Its longitudinal and lateral axes $a$ and $b$ can be found in the following way; we omit the details.

From the radii $r, r_y$ and $r_y^*$ (3-38) we determine the quantities $A, B, \rho$, and $\tau$ by means of the formulae

$$A = 1/2r, \quad B = 1/2r_y - 1/2r_y^*, \tag{6-1}$$

$$\rho^{-1} = (A+B)/2, \tag{6-2}$$

$$\tau = \arccos\left(\frac{|A-B|}{A+B}\right) \; . \tag{6-3}$$

Then the quantities $e$ and $g$ are found from the transcendental equation

$$\frac{e^2(\underline{D}-\underline{C})}{\underline{E}} = \cos\tau \; , \tag{6-4}$$

where the complete elliptic integrals $\underline{C}, \underline{D}$ and $\underline{E}$ are functions of $b$:

$$\underline{C} = \int_0^{\pi/2} \frac{\cos^2\theta\sin^2\theta\, d\theta}{(1-e^2\sin^2\theta)^{3/2}}, \quad \underline{D} = \int_0^{\pi/2} \frac{\sin^2\theta\, d\theta}{(1-e^2\sin^2\theta)^{1/2}}, \tag{6-5a-b}$$

$$\underline{E} = \int_0^{\pi/2} (1-e^2\sin^2\theta)^{1/2}\, d\theta, \tag{6-5c}$$

$$g = \sqrt{1-e^2} \; . \tag{6-6}$$

This enables us to find the axis ratio $a/b$:

$$a/b = e \quad for\ A \geq B, \\ = 1/e\ for\ A \leq B, \tag{6-7}$$

whereas the quantity

$$c = \sqrt{ab} \tag{6-8}$$

is determined by

$$c = \sqrt{\frac{3(1-\nu)N\rho \underline{\underline{E}}}{4\pi G\sqrt{g}}}\ , \tag{6-9}$$

$\nu$ *(Greek letter nu)* being the contraction coefficient and $G$ the shear modulus (the latter notation being used only in the present section).

When the elastic constants for the two contacting bodies are equal, the normal contact does not influence the tangential contact vice versa. For the latter contact we first have to calculate the creep and spin quantities

$$\upsilon_x = W_{tx}/V, \quad \upsilon_y = W_{ty}/V, \quad \varphi = \omega_n/V, \tag{6-10}$$

by means of (5-6) and (5-7). Then we put

$$\upsilon = \sqrt{\upsilon_x^2 + \upsilon_y^2}\ . \tag{6-11}$$

Next we introduce the "reduced creep" and the "reduced spin" by

$$\zeta = \rho\upsilon/\mu c, \quad \chi = \rho\varphi/\mu \tag{6-12}$$

and the creep angle $\alpha$ by

$$\cos\alpha = \upsilon_x/\upsilon, \quad \sin\alpha = \upsilon_y/\upsilon\ ; \tag{6-13}$$

moreover, the "reduced tangential force" by

$$f_x = T_x/\mu N, \quad f_y = T_y/\mu N\ . \tag{6-14}$$

Here $\mu$ is the coefficient of dry friction.

The Kalker creep law enunciates that $f_x$ and $f_y$ only depend on the four reduced quantities $a/b, \alpha, \zeta$ and $\chi$:

$$f_x = f_x(a/b, \alpha, \zeta, \chi), \quad f_y = f_y(a/b, \alpha, \zeta, \chi)\ . \tag{6-15}$$

They can be determined by means of Kalker's accurate programs DUVOROL and CONTACT and his approximate program FASTSIM; the determination of $f_x$ and $f_y$ can be considerably accelerated by first preparing tables for these quantities.

## 7. The equations of motion

The motion of the wheelset is determined by the differential equation (4-20), the algebraic equation (4-35) and the creep law equations (6-15). The equation system can be described in the following way.

We first introduce the vectors

$$\underline{x}=[u,\chi,v,\psi]^T,\quad \underline{y}=\underline{x}',\quad \underline{z}=\begin{bmatrix}\underline{x}\\ \underline{y}\end{bmatrix}, \tag{7-1}$$

$$\underline{t}=[T_{x1},T_{y1},T_{x2},T_{y2}]^T,\quad n=[N_1,N_2],\quad \underline{f}=\begin{bmatrix}\underline{t}\\ \underline{n}\end{bmatrix}. \tag{7-2}$$

Then (4-20) reads schematically:

$$\underline{z}'=\underline{g}(\underline{z},\underline{f})\ , \tag{7-3}$$

whereas we can combine the 2 equations (4-35) for $N_1,N_2$ and the 4 creep law equations (6-15) to

$$\underline{f}=\underline{f}(\underline{z})\ . \tag{7-4}$$

When the 4 coordinates and the 4 derivatives are given, we can determine the 6 forces $T_{xj},T_{yj},N_j$ by means of (7-4): this equation can be solved iteratively by starting each iteration with the values of the 4 tangential forces belonging to the previous integration step. Then (7-3) is solved numerically by means of the Runge-Kutta-Fehlberg 23 method. Further details are given in the explication of the MatLab program scwlstnl.m

Note that the above-mentioned procedure allows us to find the values of the tangential and normal forces already at the beginning of the first integration step, which often is useful for plotting the numerical results.

## 8. Linearization

We have found that it is quite possible to linearize the equations of motion in an exact way for the case that the parasitic motion is so small that the variations of the conicities, too, remain small.

In the linear case the four equations of motion (7-3) break up into two sets of two equations: one set for the "symmetric motion" ($u$ and $\chi$) and one for the "lateral motion" ($v$ and $\psi$). For the linear equations a separate integration program has been set up: scwlsetl.m For small initial values of the displacements the results of the linear and the non-linear programs are almost identical.

## 9. The case of double-point contact

In sec. 3 we already mentioned that for certain combinations of the rail and the wheel profiles double-point contact is possible; we there indicated how this problem can be tackled as far as the geometric aspect is concerned.

The dynamics of the problem are rather complicated. In fact, when double-point contact arises, our model will show a discontinuous change of the lateral wheelset velocity, resulting in a collision and a momentary infinite value of the normal force in the additional contact point.

We think that a solution of the problem can be found by adding a certain lateral elasticity to the rail, and to suppose that it is divested from mass. Then during the contact in a second contact point the force remains finite. There is a complication because now the number of degrees of freedom of the system is larger. However, it seems that this complication is surmountable.

An investigation on the dynamics of the double-point contact is nowadays performed in Delft and in Dnepropetrovsk simultaneously.

## 10. Concluding remarks

As we already mentioned in the introduction, we restricted the description of the theory to the case of a tangent track, the vehicle body motion being purely translational. It goes without saying that the investigation of the wheelset motion on a curved track or on an irregular track is not very meaningful when the parasitic motion of the vehicle body is not considered at all. However, an adequate extension of the model by taking this parasitic motion into account does not give rise to important complications. Publications on such extended models will appear in due time.

## Literature

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2. Seydel, R., From equilibrium to chaos, Elsevier (1988), 11+367 pp., ISBN 0-444-01250-8.

3. Kalker, J.J., On the rolling contact of two elastic bodies in the presence of dry friction, Thesis Delft (1967), 7+155 pp.

4. Kalker, J.J., Three-Dimensional Elastic Bodies in Rolling Contact, Dordrecht (1990), 26+314 pp., ISBN 0-7923-0712-7.

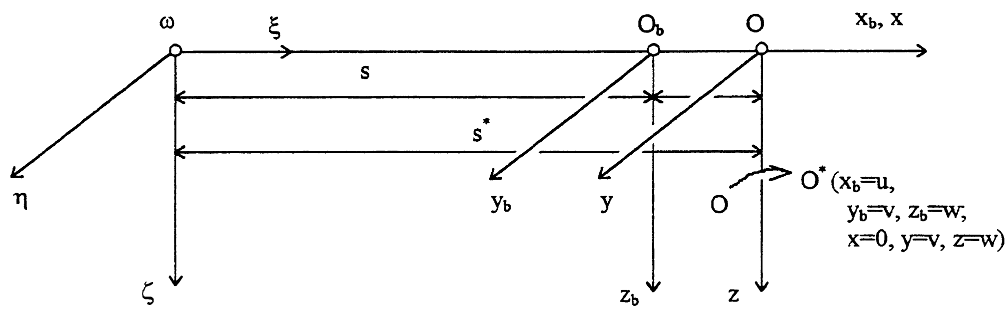


Fig. 2-1. The position of the wheelset and the vehicle body

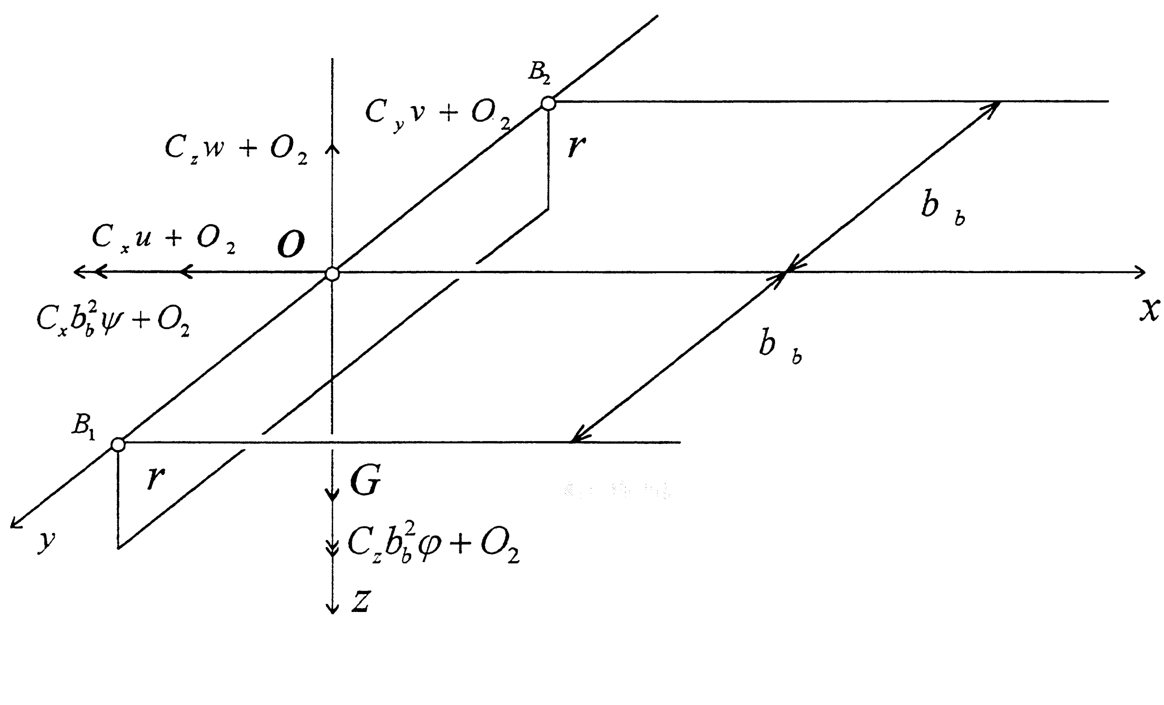


Fig. 2-2. The forces which are applied at the wheelset

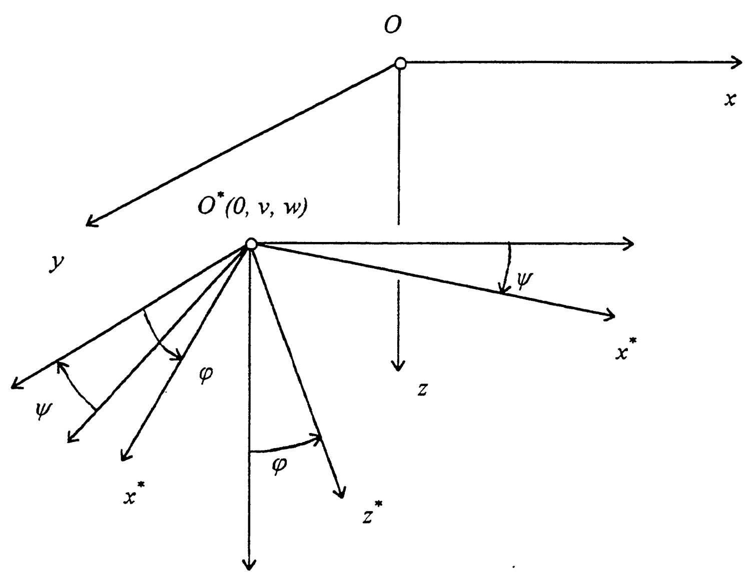


Fig. 3-1. The displacement of the wheelset

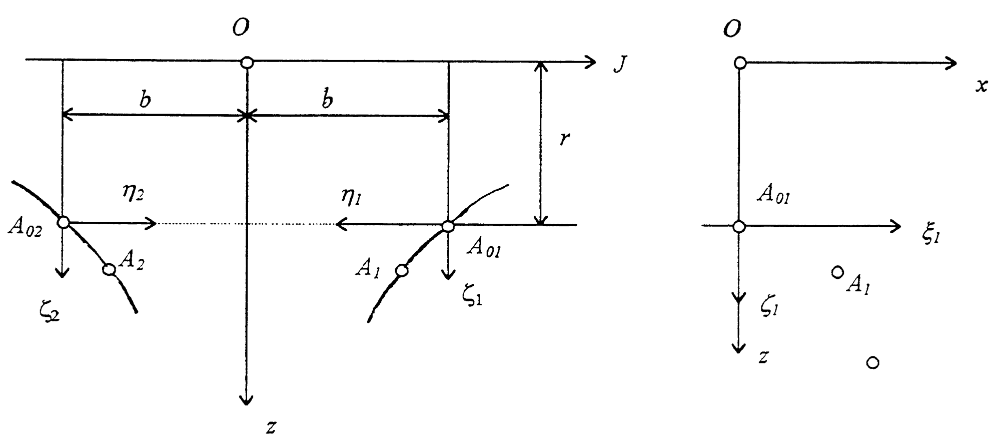


Fig. 3-2. The rail surface

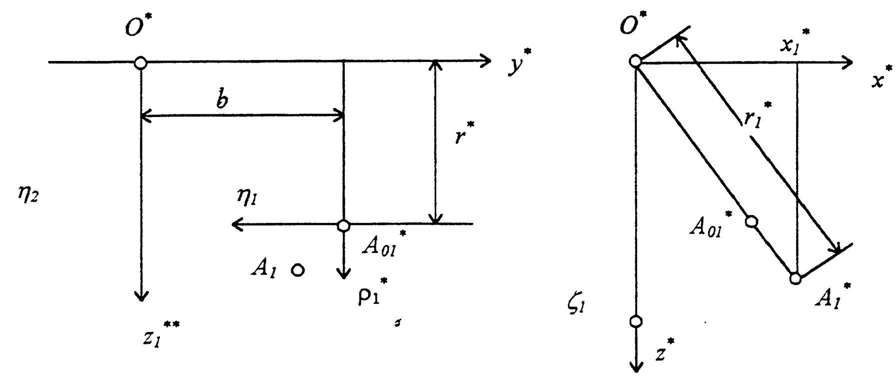


Fig. 3-3. The surface of the right-hand wheel

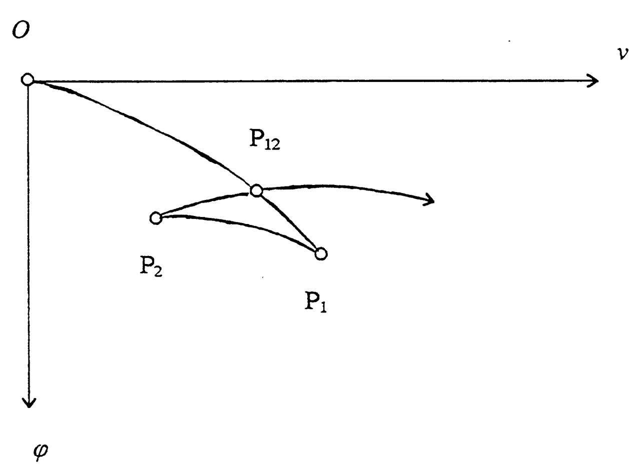


Fig. 3-4. The relation between $v$ and $\varphi$ in the neighbourhood of a double-contact point

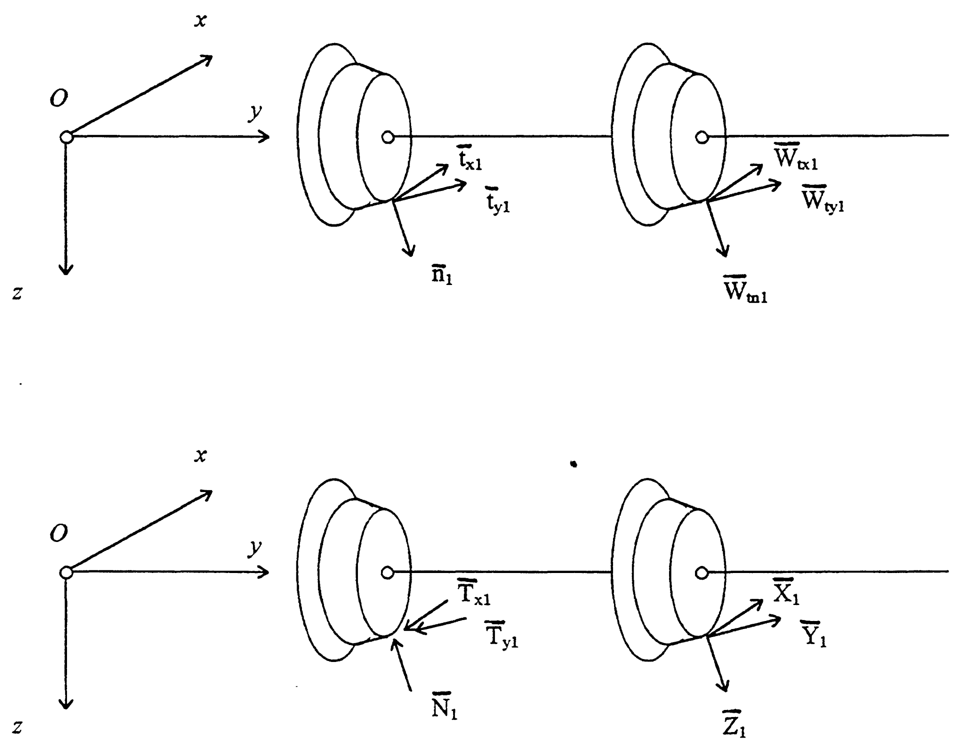


Fig. 4-1. The velocities and forces in the right hand contact point