Efficient Calculation of Transverse Stresses in Composite Plates

Raimund Rolfes*, Ahmed K. Noor**, Klaus Rohwer*

*Institute of Structural Mechanics, DLR, Braunschweig, Germany
**Center for Advanced Computational Technology, University of Virginia,
NASA Langley Research Center, Hampton, VA

ABSTRACT

Transverse stresses play an important role in the onset and growth of damage in composite structures. A post-processing method is presented which provides transverse shear and normal stresses in composite plates subjected to mechanical and thermal loads. The analytical formulation is based on the first-oder shear deformation theory and the plate is descretized by using a single-field displacement finite element model. The procedure is based on neglecting the derivatives of the in-plane forces and the twisting moments, as well as the mixed derivatives of the bending moments, with respect to the in-plane coordinates. The method is easily adapted to commercial FE-codes like MSC/NASTRAN.

1 Introduction

Fiber composite technology offers a very significant potential for weight saving of many aerospace structures. That´s why it is increasingly used by aerospace industry and replaces conventional metallic structures. An illustrative example is the outer wing for the Airbus A3XX which is currently under development in Germany. However, there are still some important obstacles remaining which prevent the technology from even faster spreading into the market.

Possibly the most important one is the very complicated failure behaviour of these inhomogeneous and anisotropic structures. It has been analytically tackled by the so-called damage mechanics of composites which was originally destined to provide a counterpart to the well-developed fracture mechanics for metallic materials. Due to the complicated interaction of different damage mechanisms, e.g. fiber-matrix debonding, matrix cracking, fibre cracking, delamination etc., no generally applicable methods for the prediction of damage progression and of residual life time could be established so far.

Even the onset of failure is insufficiently described by most of the failure criteria which are implemented into common finite element (FE) packages. Presently, MSC/NASTRAN offers the criteria of Hill, Hoffman, Tsai-Wu and the maximum strain method. The first three do not distinguish between fiber and matrix breakage. However, this information is very important since many kinds of matrix cracks are tolerable whereas rupture of fibers can cause immediate breakdown of the whole structure. Moreover, all of the the aforementioned methods base on the assumption of a plane state of stress. In thin-walled structures the transverse stresses are clearly much smaller than the in-plane stresses. Nevertheless, they can decisively influence the onset and growth of delaminations. Therefore, modern failure criteria (see e.g. [1], [2]) which account for all six stress components should be implemented into MSC/NASTRAN.

In order to provide the required input data for such an improved failure prediction an efficient method for calculating transverse stresses in composite structures must be developed. Various techniques have been proposed for the accurate determination of transverse stresses in laminated composites. These include using 1) three-dimensional, or quasi-three-dimensional finite elements (see, for example [3], [4], [5], [6]); 2) two-dimensional finite elements based on higher-order shear deformation theories with either nonlinear or piecewise linear approximations for the displacements in thickness direction (see, for example, [7], [8], [9]); and 3) post-processing techniques used in conjunction with two-dimensional finite elements based on the classical or first-order shear deformation theory (with linear displacement approximation through the thickness of the entire laminate).

Experience with most of the three-dimensional finite elements and two-dimensional finite elements based on higher-order shear deformation theories, has shown that unless the three-dimensional equilibrium equations are used in evaluating the thickness distribution of the transverse stresses, the resulting stresses are inaccurate (see, for example, [8], [10], [11], [12], [13]). Since the finite element models based on the first-order shear deformation theory are considerably less expensive than those based on three-dimensional and higher-order two-dimensional theories, their use in conjunction with post-processing techniques has received increasing attention in recent years. The post-processing techniques proposed for the evaluation of transverse stresses are based on the use of a) three-dimensional equilibrium equations (see [12], [14], [15], [16]); b) predictor-corrector approaches (see [17]) and c) use of simplifying assumptions ([18], [19]).

Except for Noor et al. [16] which considered only transverse shear stresses, none of the cited references considered transverse stresses in thermally loaded laminates. The present study focuses on the accurate evaluation of both transverse shear and transverse normal stresses, in composite panels subjected to mechanical and thermal loads. The post-processing technique, based on the use of simplifying assumptions and presented in [18] and [19] is extended herein to the case of thermal stresses. The effectiveness of the proposed procedure is demonstrated by means of numerical examples of cross-ply panels.

2 Theory

Basic Idea

The present method has in common with other postprocessing techniques (e.g. [12], [14], [15], [16]), the use of three-dimensional equilibrium conditions to calculate the transverse stresses using the derivatives of the in-plane stresses. However, in contradistinction to other approaches the transverse shear stresses are expressed by the shear forces and the first derivatives of the temperature field with respect to the in-plane coordinates only. This results in saving one order of differentiation of the shape functions compared to other methods. The idea goes back to Rohwer [20] who introduced the following two simplifying:

1) the effect of the in-plane stress resultants on the transverse shear stresses is neglected, and

2) a cylindrical bending mode is assumed in each direction

Because of the reduction in the order of differentiation, the present method can, in many cases, provide a good approximation of the transverse normal stress using only eight-noded finite elements.

Transverse Shear Stresses

The FSDT bases on the subsequent assumptions:

1) The laminates are composed of a number of perfectly bonded layers

2) The strains are linear in the thickness direction, i.e.

$$\underline{\varepsilon} = \underline{\varepsilon}^{0} + x_3 \underline{\kappa} \qquad (1)$$

where $\underline{\varepsilon}^{0}$ and $\underline{\kappa}$ are the extensional strains and curvature changes of the middle surface and x_3 is the thickness coordinate.

3) Every point of the laminate is assumed to possess a single plane of thermoelastic symmetry parallel to the middle plane (monoclinic symmetry).

4) The material properties are independent of the temperature.

5) The in-plane stresses $\underline{\sigma}_m$ and strains $\underline{\varepsilon}$ are related by the plane stress constitutive relation

$$\underline{\sigma}_m = \underline{\overline{C}}\left(\underline{\varepsilon} - \underline{\alpha}\Delta T\right) \qquad (2)$$

where $\underline{\overline{C}}$ is the plane-stress stiffness matrix, $\underline{\alpha}$ is the vector of the coefficients of thermal expansion, and ΔT is the temperature change.

It is worthwile to note that the stiffness matrix $\underline{\overline{C}}$ has been derived under the assumption of vanishing transverse normal stress. Nevertheless, this stress component will be evaluated later on by use of 3D equilibrium conditions.

Introducing Eq. (1) into Eq. (2) results in

$$\underline{\sigma}_m = \underline{\overline{C}}\left(\underline{\varepsilon}^{0} + x_3\underline{\kappa} - \underline{\alpha}\Delta T\right) \qquad (3)$$

This is the standard equation for in-plane stress recovery which is usually implemented into commercial FE-packages offering composite analysis on the basis of the FSDT. The present methodology for calculating the full state of stress is based on the following five sets of equations:

1) 3D Equilibrium Equations

$$\sigma_{13,3} + \sigma_{11,1} + \sigma_{12,2} = 0 \qquad (4)$$

$$\sigma_{23,3} + \sigma_{22,1} + \sigma_{12,1} = 0 \qquad (5)$$

$$\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} = 0 \qquad (6)$$

2) Simplifying Assumptions

$$\underline{N}_{,\alpha} = \underline{0} \tag{7}$$

$$M_{11,2} = M_{22,1} = M_{12,1} = M_{12,2} = 0 \tag{8}$$

3) Material Law for Transverse Shear

$$\underline{Q} = \underline{H}\underline{\gamma} \tag{9}$$

4) Equilibrium Conditions for the Laminate after Introduction of Simplifying Assumptions

$$M_{11,1} = Q_1 \tag{10}$$

$$M_{22,2} = Q_2 \tag{11}$$

5) Thermoelastic Constitutive Relation for the Laminate

$$\begin{bmatrix} \underline{N} \\ \underline{M} \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{B} & \underline{D} \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}^{0} \\ \underline{\kappa} \end{bmatrix} - \begin{bmatrix} \underline{N}^{th} \\ \underline{M}^{th} \end{bmatrix} \tag{12}$$

with

$$\begin{bmatrix} \underline{N}^{th} \\ \underline{M}^{th} \end{bmatrix} = \int_h \begin{bmatrix} \underline{\overline{C}}\,\underline{\alpha}\,\Delta T \\ \underline{\overline{C}}\,\underline{\alpha}\,x_3\,\Delta T \end{bmatrix} dx_3 \ . \tag{13}$$

$\underline{N}$, $\underline{Q}$ and $\underline{M}$ are the extensional, transverse shear and bending stress resultants, and $\underline{\gamma}$ constitutes the transverse shear strains, respectively. $\underline{H}$, $\underline{A}$, $\underline{D}$ and $\underline{B}$ are the transverse shear, extensional, bending and bending-extensional coupling stiffness matrices of the laminate, respectively. The conventional equilibrium approach (e.g. [12], [14], [15], [16]) uses only the 3D equilibrium conditions (Eqs. (4)-(6)). Introducing the stress recovery equation (3) into the equlibrium conditions (4) and (5) and resolving with respect to the transverse shear stresses yields

$$\underline{\tau}(x_3) = \begin{bmatrix} \sigma_{31} \\ \sigma_{32} \end{bmatrix} = -\int_{\varsigma=-\frac{h}{2}}^{x_3} \underline{B}_{\alpha}\,\underline{\overline{C}}\left(\underline{\varepsilon}_{,\alpha}^{0} + x_3\,\underline{\kappa}_{,\alpha} - \underline{\alpha}\,\Delta T_{,\alpha}\right) d\varsigma \ . \tag{14}$$

Although a matrix formulation is adopted, the Einstein summation convention shall be applied, and the range of the Greek subscripts is 1,2. $\underline{B}_{\alpha}$ are the Boolean matrices

$$\underline{B}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

$$\underline{B}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \qquad (16)$$

Eq. (14) is used for the standard equilibrium approach. Although applicable to finite elements, it has a major drawback. Regarding typical finite plate elements based on the FSDT with five variables per node, equivalent shape functions are introduced for the displacements (u_0 , v_0 , w) and the rotations (φ_x , φ_y). Then, the membrane strains and curvatures need first derivatives of the shape functions for u_0 , v_0 , φ_x , φ_y , and, consequently, second order derivatives are required for Eq. (14). Since one more differentiation is necessary for the transverse normal stress calculation, at least cubic shape functions must be chosen for u_0 , v_0 , φ_x , φ_y , if all derivatives shall be calculated on the element level. This is shown in column 2 of Table 1.

Table 1. The requirements can be lowered to some extent if the influence of the derivatives of the in-plane displacements u_0 and v_0 on the transverse normal stress is neglected. Then, linear shape functions suffice for the in-plane displacements, whereas cubic ones remain necessary for the rotations. This method can be denoted as a reduced equilibrium approach and is indicated in column 3 of Table 1.

Table 1: Degree of polynomials required for the calculation of the transverse shear stresses and the transverse normal stress (values in brackets) on element level

functional dof´s	degree of polynomials required for calculation of transverse stresses		
	equilibrium method based on eq. (4)		present method
	full	reduced	
u_0	2 (3)	1 (1)	1 (1)
v_0	2 (3)	1 (1)	1 (1)
w_0	1 (1)	1 (1)	1 (2)
φ_x	2 (3)	2 (3)	1 (1)
φ_y	2 (3)	2 (3)	1 (1)

The authors suggest a method that puts significantly lower requirements on the polynomial order of the shape functions (conf. column 4 of Table 1). The sequence of calculation steps is as follows.

Firstly, the thermoelastic constitutive relation for the laminate (Eq. (12))is resolved with respect to $\underline{\varepsilon}^0$ and $\underline{\kappa}$. This yields

$$\begin{bmatrix} \underline{\varepsilon}^0 \\ \underline{\kappa} \end{bmatrix} = \begin{bmatrix} \underline{\tilde{A}} & \underline{\tilde{B}} \\ \underline{\tilde{B}} & \underline{\tilde{D}} \end{bmatrix} \left(\begin{bmatrix} \underline{N} + \underline{N}^{th} \\ \underline{M} + \underline{M}^{th} \end{bmatrix} \right) \qquad (17)$$

where $\underline{\tilde{A}}$, $\underline{\tilde{B}}$ and $\underline{\tilde{D}}$ are the compliance matrices of the panel (inverse of the panel stiffness matrices). If the temperature has a linear variation through the thickness, i.e.

$$\Delta T = T^0 + x_3 T^1 \tag{18}$$

the thermal stress resultants appear as

$$\begin{bmatrix} \underline{N}^{th} \\ \underline{M}^{th} \end{bmatrix} = \begin{bmatrix} \underline{A}^{th} & \underline{B}^{th} \\ \underline{B}^{th} & \underline{D}^{th} \end{bmatrix} \begin{bmatrix} T^0 \\ T^1 \end{bmatrix} \tag{19}$$

where

$$\begin{bmatrix} \underline{A}^{th} \\ \underline{B}^{th} \\ \underline{D}^{th} \end{bmatrix} = \int_h \begin{bmatrix} \underline{\overline{C}}\,\underline{\alpha} \\ \underline{\overline{C}}\,\underline{\alpha}\; x_3 \\ \underline{\overline{C}}\,\underline{\alpha}\; x_3^2 \end{bmatrix} dx_3 \; . \tag{20}$$

Introducing Eq.´s (17) and (18) into Eq. (14) gives an expression for the transverse shear stresses depending on the stress resultants and the temperature field,

$$\underline{\tau} = \underline{B}_\alpha \left(\underline{F}\left(\underline{M},_\alpha + \underline{M},_\alpha^{th}\right) + \underline{G}\left(\underline{N},_\alpha + \underline{N},_\alpha^{th}\right) + \underline{a}^{th} T,_\alpha^0 + \underline{b}^{th} T,_\alpha^1 \right) \tag{21}$$

where

$$\begin{bmatrix} \underline{G} \\ \underline{F} \end{bmatrix} = -\int_{\varsigma=-\frac{h}{2}}^{x_3} \begin{bmatrix} \underline{\overline{C}}\left(\underline{\tilde{A}} + \varsigma\,\underline{\tilde{B}}\right) \\ \underline{\overline{C}}\left(\underline{\tilde{B}} + \varsigma\,\underline{\tilde{D}}\right) \end{bmatrix} d\varsigma \tag{22}$$

and

$$\begin{bmatrix} \underline{a}^{th} \\ \underline{b}^{th} \end{bmatrix} = \int_{\varsigma=-\frac{h}{2}}^{x_3} \begin{bmatrix} \underline{\overline{C}}\,\underline{\alpha} \\ \underline{\overline{C}}\,\underline{\alpha}\; x_3 \end{bmatrix} d\varsigma \; . \tag{23}$$

The quantities $\underline{a}^{th}$ and $\underline{b}^{th}$ are integrals over part of the laminate thickness of products of material stiffnesses and coefficients of thermal expansion. Therefore, they can be thought of as partial thermal stiffnesses. Now the simplifying assumptions (Eqs. (7) and (8)) are introduced. Then, the terms $\underline{N},_\alpha$ in Eq. (21) vanish and the remaining derivatives of the bending stress resultants can be expressed by the transverse shear forces (Eqs. (10) and (11)). One finally gets

$$\underline{\tau} = \underline{f}\underline{Q} + \underline{B}_\alpha \left(\begin{bmatrix} \underline{G} & \underline{F} \end{bmatrix} \begin{bmatrix} \underline{A}^{th} & \underline{B}^{th} \\ \underline{B}^{th} & \underline{D}^{th} \end{bmatrix} + \begin{bmatrix} \underline{a}^{th} & \underline{b}^{th} \end{bmatrix} \right) \begin{bmatrix} T,_\alpha^0 \\ T,_\alpha^1 \end{bmatrix} \tag{24}$$

where $\underline{f}$ are the components of $\underline{F}$ multiplying the remaining derivatives of $\underline{M}$, i.e.

$$\underline{f} = \begin{bmatrix} F_{11} & F_{32} \\ F_{31} & F_{22} \end{bmatrix} , \tag{25}$$

and Eq. (19) has been used. The material law for transverse shear (Eq. (9)) is used to evaluate the transverse shear forces. Thus, the present procedure is not a pure equilibrium method, but a mixture between material law and equilibrium approach.

Transverse Normal Stress

By using the third equilibrium condition the foregoing methodology can be applied to the evaluation of the transverse normal stress, despite this stress component has been neglected previously. Introducing Eq. (24) into Eq. (6) and resolving with respect to σ_{33} gives

$$\sigma_{33} = -\underline{b}_\alpha \underline{f}^* \underline{Q}_{,\alpha} + \underline{b}_\alpha \underline{B}_\beta \left(\begin{bmatrix} \underline{G}^* & \underline{F}^* \end{bmatrix} \begin{bmatrix} \underline{A}^{th} & \underline{B}^{th} \\ \underline{B}^{th} & \underline{D}^{th} \end{bmatrix} + \begin{bmatrix} \underline{a}^{*th} & \underline{b}^{*th} \end{bmatrix} \right) \begin{bmatrix} T^{0}_{,\alpha\beta} \\ T^{1}_{,\alpha\beta} \end{bmatrix} \tag{26}$$

where

$$\underline{b}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} , \quad \underline{b}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad (27)(28)$$

and

$$\begin{bmatrix} \underline{f}^* \\ \underline{G}^* \\ \underline{F}^* \\ \underline{a}^{*th} \\ \underline{b}^{*th} \end{bmatrix} = \int_{\varsigma=-\frac{h}{2}}^{x_3} \begin{bmatrix} \underline{f} \\ \underline{G} \\ \underline{F} \\ \underline{a}^{th} \\ \underline{b}^{th} \end{bmatrix} d\varsigma \ . \tag{29}$$

It should be noted that the boundary conditions at the top and bottom surfaces are automatically satisfied in the present procedure. This applies to the transverse shear as well as to the transverse normal stresses and has been shown in [24].

3 Implementation

The present method is implemented into the finite element program B2000 and the post-processor TRAVEST. B2000 is a common research tool for a number of organizations, including NLR, CIRA, SMR, DLR and some universities in the Netherland and Switzerland. Within B2000 a standard isoparametric eight node element with reduced integration (2x2) is used. The same shape functions are used for interpolating the temperature distribution. Since the evaluation of transverse stresses requires the first derivative of the transverse shear forces ($\underline{Q}_{,\alpha}$) and the second derivatives of the temperature field ($T^{0}_{,\alpha\beta}$, $T^{1}_{,\alpha\beta}$), then second order derivatives of the shape functions are required. The derivatives can be evaluated using the procedure outlined in [11]. The values for $\underline{Q}_{,\alpha}$, $T^{0}_{,\alpha\beta}$ and $T^{1}_{,\alpha\beta}$ are input to the post-processor TRAVEST which calculates all quantities that depend on the transverse coordinate x_3 only, i.e. the matrices $\underline{f}, \underline{G}, \underline{F}, \underline{a}^{th}, \underline{b}^{th}, \underline{A}^{th}, \underline{B}^{th}$ and $\underline{D}^{th}$, and its integrals over x_3 which are denoted by stars. These integrations are carried out analytically. Thus, no additional numerical errors are introduced. If the material properties of each layer are uniform (i.e., independent of x_α and x_3), the additional numerical effort is small since the aforementioned matrices have to be evaluated only once. The multiplications with $\underline{Q}_{,\alpha}$, $T^{0}_{,\alpha\beta}$ and $T^{1}_{,\alpha\beta}$ (according to Eqs. (24) and (26) must be performed at each point where the transverse stresses are required.

TRAVEST can also be used as a postprocessor to commercial finite element codes like e.g. MSC/NASTRAN.

4 Numerical Examples

To assess the effectiveness of the foregoing postprocessing procedure for calculating exact transverse stresses in composite plates several multilayered composite panels were analyzed. The panels were subjected to transverse static loading or to a thermal loading, in the form of either a constant temperature change or a temperature gradient in the thickness direction. Typical results are presented herein for a ten-layered symmetric cross-ply laminate ($[0/90/0/90/0]_{sym}$) and a four-layered antisymmetric laminate ([0/90/0/90]), with the fibers of the top layer parallel to the x_1 axis. The first laminate exhibits no bending-extensional coupling, and the second laminate shows a strong coupling. Two aspect ratios were selected; namely, $L_2/L_1 = 1$ and 2. Furthermore, two different thickness ratios, $h/L_1 = 0.05$ and 0.1, were considered. Each of the transverse loading, uniform temperature through the thickness, and the temperature gradient through-the-thickness, had a double sinusoidal variation in the x_1 - x_2 plane. The amplitudes of the trasnverse loading, uniform temperature and temperature gradient (p^0, T^0, T^1) were chosen to be one. The following boundary conditions, which allow an exact three-dimensional solution to be obtained ([16], [17], [23]), were selected:

$$u_2 = 0,\ w = 0,\ \varphi_2 = 0,\ \sigma_{11} = 0 \qquad \text{at} \qquad x_1 = 0,\ L_1$$
$$u_1 = 0,\ w = 0,\ \varphi_1 = 0,\ \sigma_{22} = 0 \qquad \text{at} \qquad x_2 = 0,\ L_2\ .$$

The three-dimensional solution was used as the standard for comparison. The material properties of the individual layers were taken to be those typical of high-modulus fibrous composites, namely:

$$E_L / E_T = 15,\ G_{LT} / E_T = 0.5,\ G_{TT} / E_T = 0.3378,\ \nu_{LT} = 0.3,\ \nu_{TT} = 0.48$$
$$\alpha_L = 0.139 \times 10^{-6},\ \alpha_T = 9 \times 10^{-6}$$

Typical results are shown in Figs. 1-3 for the transverse shear stresses and in Figs. 4 and 5 for the transverse normal stress. The results are discussed subsequently.

Effect of Laminate Parameters and Loading on the Magnitude and Distribution of Transverse Stresses Through the Thickness

- The transverse shear and normal stresses produced by the transverse loading p have a smooth variation in the thickness direction.
- The relative magnitudes of the transverse shear stresses, σ_{31}, σ_{32} , and of the transverse normal stress, σ_{33} , is strongly influenced by the aspect ratio of the laminate. For square laminates, σ_{31} and σ_{32} have almost equal magnitudes, but opposite signs, and σ_{33} is very small. On the other hand, for rectangular laminates, the magnitudes of σ_{31} and σ_{32} are different and the ratio of $max\ \sigma_{33} / max\ \sigma_{3\beta}$ is larger than that for square laminates (see Table 2).

Table 2: Accuracy of the Transverse Normal Stress Component σ_{33} for the Thermal Loading Cases T^0 and T^1

Laminate		Loading	Bending	$\frac{max\sigma_{33}}{max\sigma_{3\beta}}$	Accuracy of σ_{33}
n	L_2/L_1				
4	1	T^0	yes	$2.5\ 10^{-3}$	poor
	2		yes	$3.6\ 10^{-2}$	satisfactory
	1	T^1	yes	$1.5\ 10^{-2}$	good
	2		yes	$7.0\ 10^{-2}$	excellent
10	1	T^0	no	n.a.	poor
	2		no	n.a.	poor
	1	T^1	yes	$5.8\ 10^{-4}$	poor
	2		yes	$1.8\ 10^{-2}$	good

Accuracy of Transverse Shear Stresses

- For all the laminates considered, the accuracy of the transverse shear stresses, $\sigma_{3\beta}$

3β

- For rectangular plates, the larger transverse shear stresses are predicted more accurately by the foregoing procedure than the smaller ones.

Accuracy of Transverse Normal Stress

- The accuracy of the transverse normal stress predicted by the foregoing procedure is strongly dependent on the relative magnitude of the in-plane and bending stress resultants (*h* N / M), and the relative magnitudes of σ_{33} and $\sigma_{3\beta}$, which, in turn, are dependent on the loading, the lamination and the geometric parameters of the panel (see Table 2).
- For the static loading case, the accuracy of σ_{33} is, for all the panels considered, excellent. The accuracy is insensitive to variations in both (h/L_1) and (L_2/L_1).
- For the thermal loading cases, the accuracy of σ_{33} is satisfactory only when the ratio (*h* N / M) is small and the ratio ($max\ \sigma_{33}$ / $max\ \sigma_{3\beta}$) is larger than 0.01. For the case of uniform temperature through-the-thickness, the accuracy of σ_{33} is satisfactory for rectangular four-layer laminates, and not satisfactory for all the other laminates considered. For the case of temperature gradient through-the-thickness, the accuracy of σ_{33} ranged from good to excellent, except for the square ten-layer laminate, where σ_{33} was less than three orders of magnitude smaller than $\sigma_{3\beta}$.

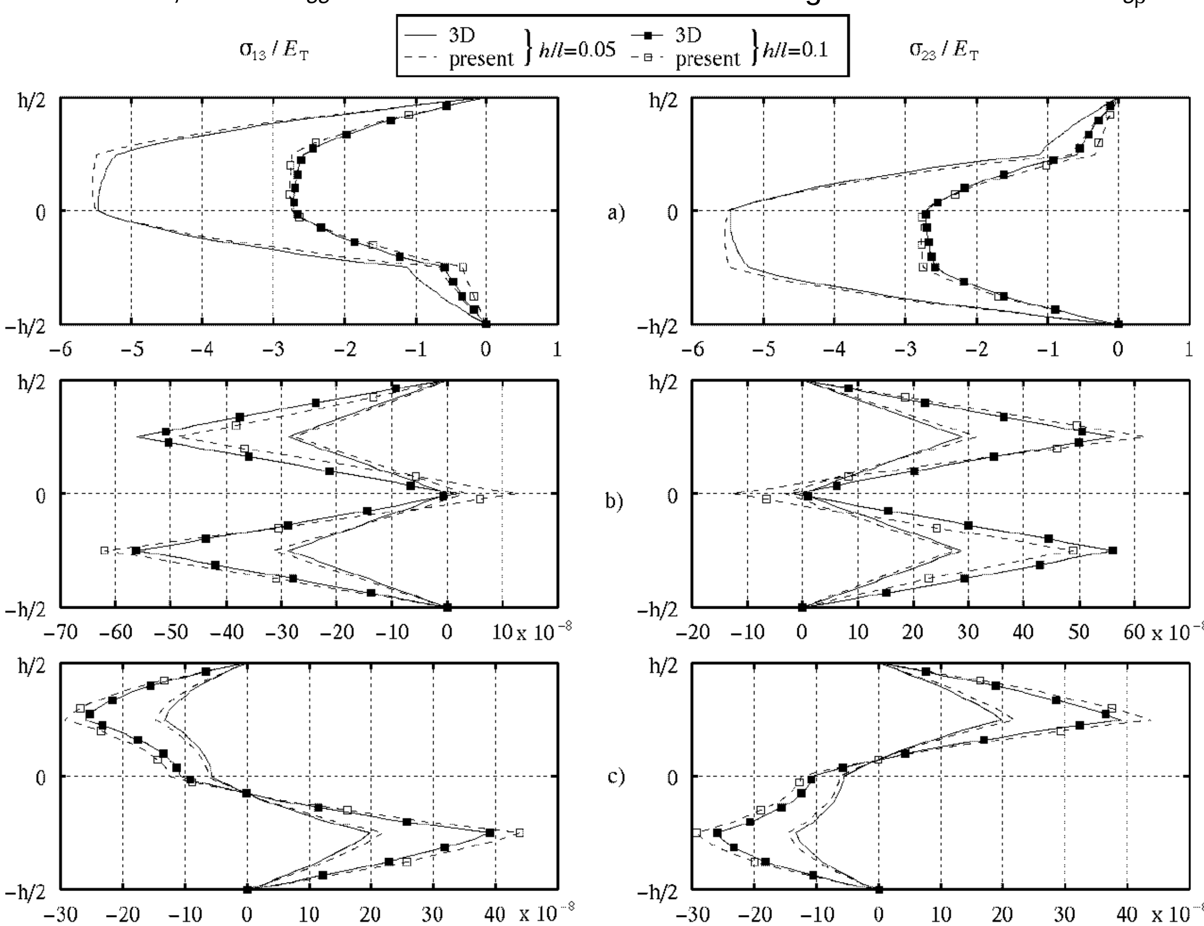

Figure 1: Through-the-thickness distributions of transverse shear stresses (σ_{31} at $(L_1,L_2/2)$;σ_{32} at $(L_1/2,L_2)$). Four-layer antisymmetric cross-ply laminate with L_2/L_1=1 subjected to a) static loading p, b) uniform temperature T^0, c) temperature gradient T^1

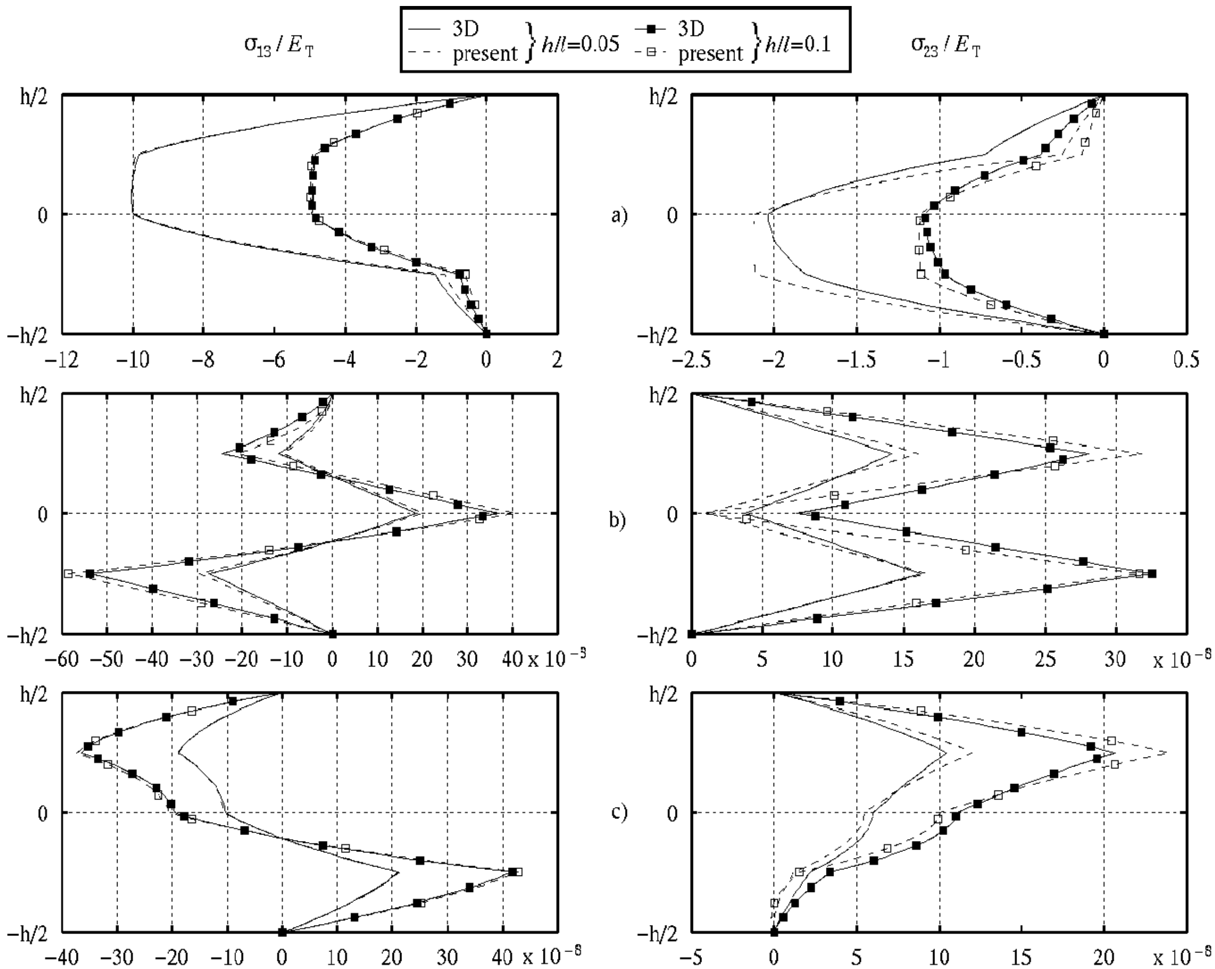

Figure 2: Through-the-thickness distributions of transverse shear stresses (σ_{31} at $(L_1,L_2/2)$;σ_{32} at $(L_1/2,L_2)$). Four-layer antisymmetric cross-ply laminate with L_2/L_1=2 subjected to a) static loading p, b) uniform temperature T^0, c) temperature gradient T^1

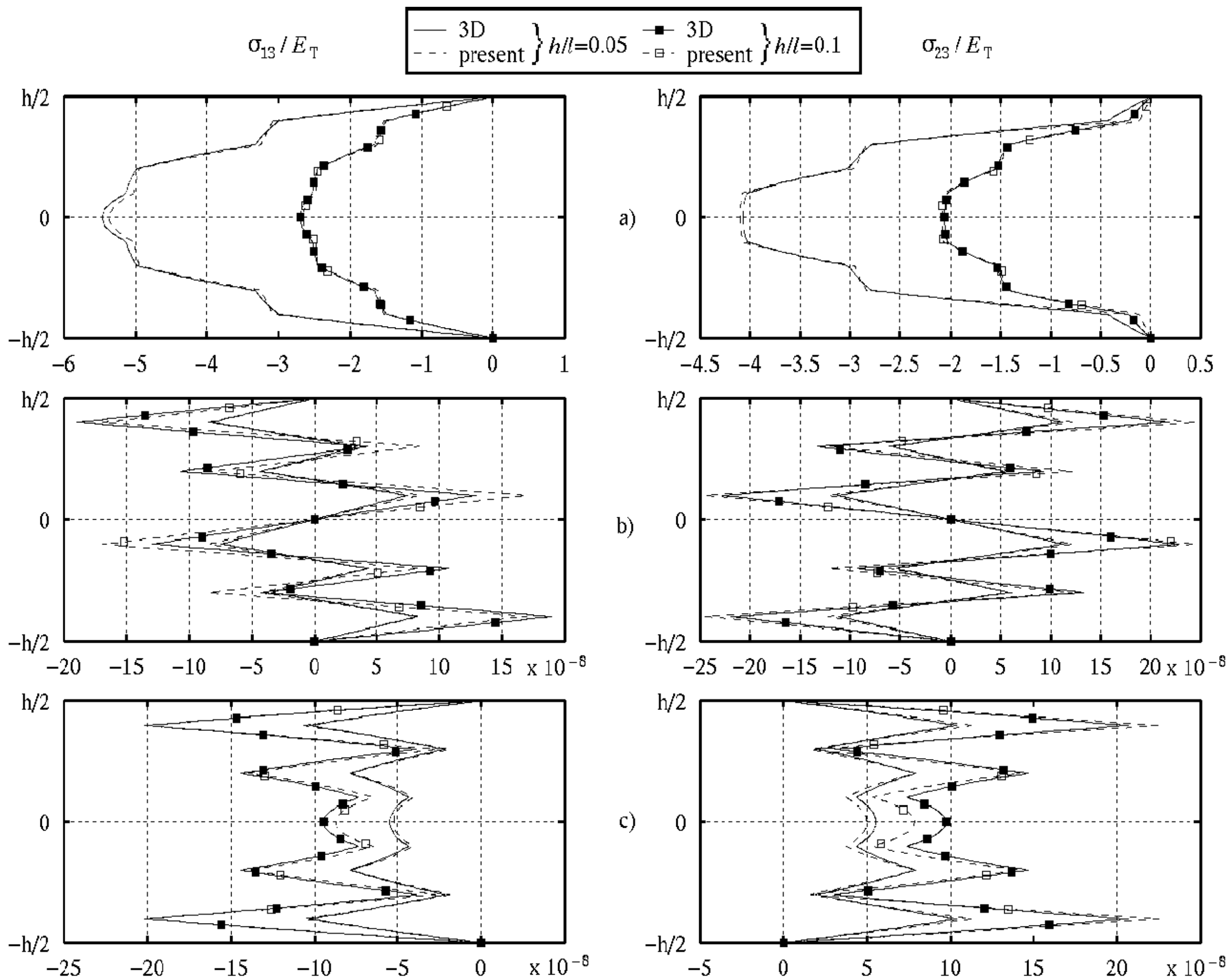

Figure 3: Through-the-thickness distributions of transverse shear stresses (σ_{31} at $(L_1,L_2/2)$; σ_{32} at $(L_1/2,L_2)$). Ten-layer symmetric cross-ply laminate with L_2/L_1=1 subjected to a) static loading p, b) uniform temperature T^0, c) temperature gradient T^1

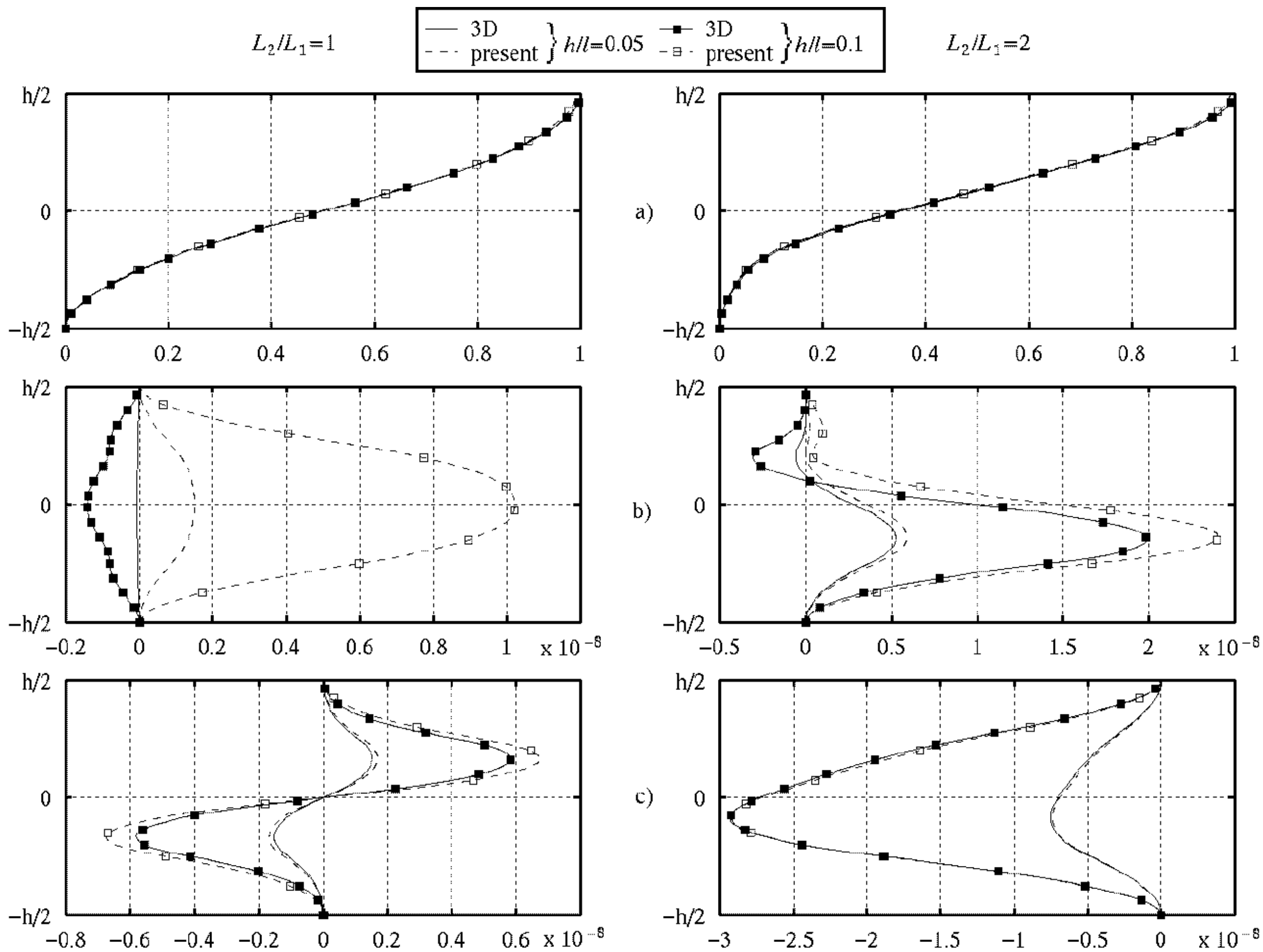

Figure 4: Through-the-thickness distribution of transverse normal stress σ_{33} at $(L_1/2, L_2/2)$. Four-layer antisymmetric cross-ply laminate subjected to a) static loading p, b) uniform temperature T^0, c) temperature gradient T^1

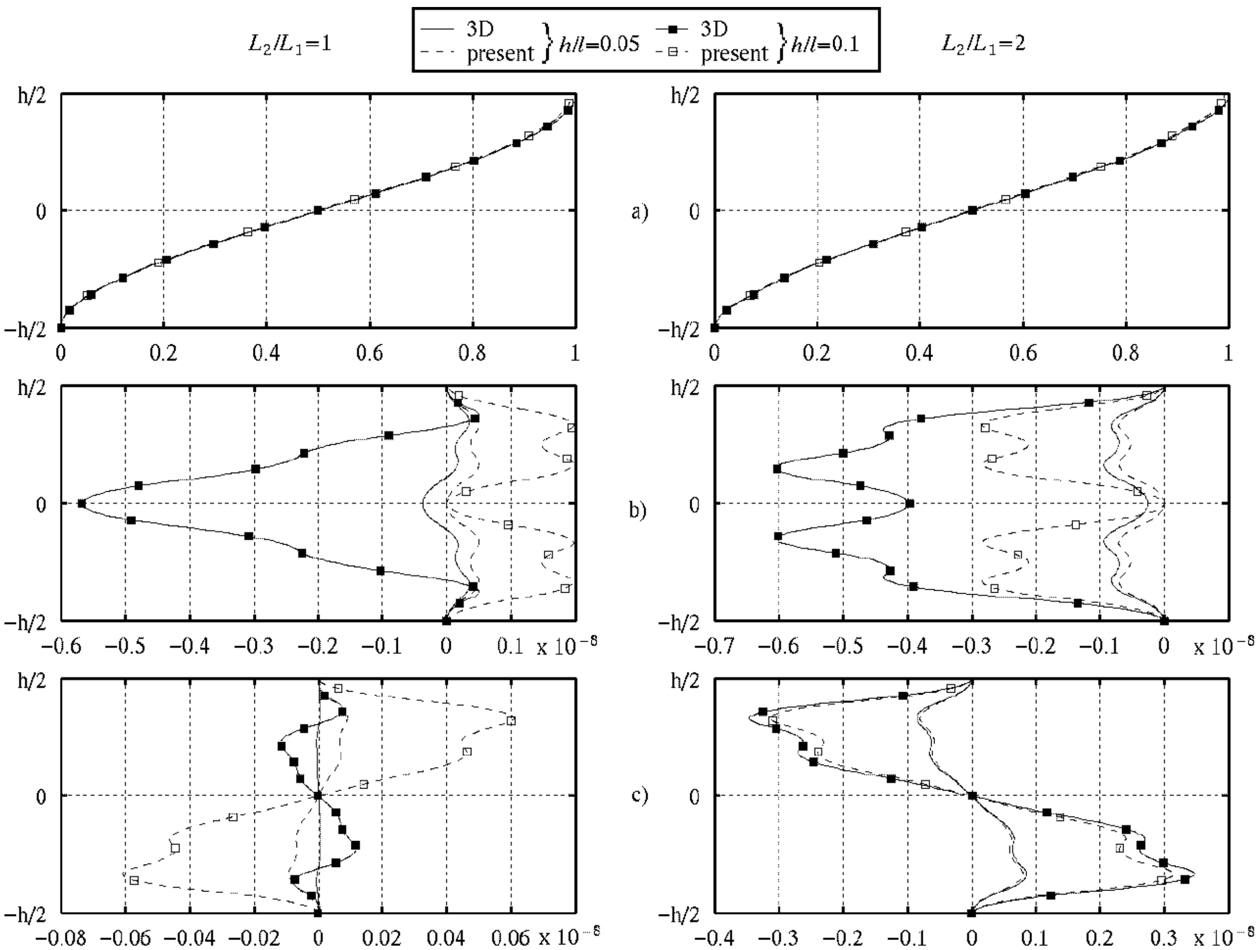

Figure 5: Through-the-thickness distribution of transverse normal stress σ $_{33}$ at (L_1/2,L_2/2). Ten-layer symmetric cross-ply laminate subjected to a) static loading p, b) uniform temperature T^0, c) temperature gradient T^1

5 References

[1] Hashin, Z.:
Failure Criteria for Unidirectional Fiber Composites, Journal of Applied Mechanics 47, 329-334 (1980)

[2] Puck, A.:
Ein Bruchkriterium gibt die Richtung an, Kunststoffe 82, 607-610 (1992).

[3] Mau, S.T.; Tong, P.; Pian, T.H.H.:
Finite Element Solution for Laminated Thick Plates, Journal of Composite Materials 6, 304-311 (1972)

[4] Spilker, R.L.; Chou, S.C.; Orringer, O.:
Alternate Hybrid Stress Elements for Analysis of Multilayer Composite Plates, Journal of Composite Materials 11, 51-70 (1977)

[5] Spilker, R.L.:
Hybrid-Stress Eight-Node Elements for Thin and Thick Multilayer Laminated Plates, International Journal for Numerical Methods in Engineering 18, 801-828 (1982)

[6] Owen, D.R.J.; Li, Z.H.:
A Refined Analysis of Laminated Plates by Finite Element Displacement Method -1: Fundamentals and Static Analysis, Computers and Structures 26, 907-914 (1987)

[7] Robbins, D.H., Jr.; Reddy, J.N.:
Modelling of Thick Composites Using a Layerwise Laminate Theory, International Journal for Numerical Methods in Engineering 36, 655-677 (1993)

[8] Chaudhuri, R.A.:
An Equilibrium Method for Prediction of Transverse Shear Stresses in a Thick Laminated Plate, Computers and Structures 23, 139-146 (1986)

[9] Gruttmann, F.; Wagner,W.:
On the Numerical Analysis of Local Effects in Composite Structures, Composite Structures 29, 1-12 (1994)

[10] Chaudhuri, R.A.; Seide,P.:
An Approximate Semi-Analytical Method for Prediction of Interlaminar Shear Stresses in an Arbitrarily Laminated Thick Plate, Computers and Structures 25, 627-636 (1987).

[11] Reddy, J.N.; Barbero, E.J.; Teply, J.L.:
A Plate Bending Element Based on a Generalized Laminate Plate Theory, International Journal for Numerical Methods in Enginerring 28, 2275-2292 (1989)

[12] Engblom, J.J.; Ochoa, O.O.:
Through-the-Thickness Stress Predictions for Laminated Plates of Advanced Composite Materials, International Journal for Numerical Methods in Enginerring 21, 1759-1776 (1985)

[13] *Kant, T.; Pandya, B.N.:*
A Simple Finite Element Formulation of a Higher-Order Theory for Unsymmetrically Laminated Composite Plates, Composite Structures 9, 215-246 (1988)

[14] *Pryor, C.W.; Barker, R.M.:*
A Finite Element Analysis Including Transverse Shear Effects for Application to Laminated Plates, AIAA Journal 9, 912-917 (1971)

[15] *Manjunatha, B.S.; Kant, T.:*
On Evaluation of Transverse Stresses in Layered Symmetric Composite and Sandwich Laminates under Flexure, Engineering Computations 10, 499-518 (1993)

[16] *Noor, A.K.; Kim, Y.H.; Peters, J.M.:*
Transverse Shear Stresses and Their Sensitivity Coefficients in Multilayered Composite Panels, AIAA Journal 32, 1259-1269 (1994)

[17] *Noor, A.K.; Burton, W.S.; Peters, J.M.:*
Predictor-Corrector Procedure for Stress and Free Vibration Analyses of Multilayered Composite Plates and Shells, Computer Methods in Applied Mechanics and Engineering 82, 341-364 (1990)

[18] *Rolfes, R.; Rohwer,K.:*
Improved Transverse Shear Stresses in Composite Finite Elements Based on First Order Shear Deformation Theory, International Journal for Numerical Methods in Engineering 40, 51-60 (1997)

[19] *Rolfes, R.; Rohwer, K.:*
Eine einfache Methode zur Ermittlung aller Querspannungen in Faserverbundplatten als Voraussetzung einer verbesserten Versagensanalyse, MSC Anwenderkonferenz, Bad Neuenahr, June 5-6 (1997)

[20] *Rohwer, K.:*
Improved Transverse Shear Stiffnesses for Layered Finite Elements, DFVLR-FB 88-32, Braunschweig (1988).

[21] *Noor, A.K.; Burton, W.S.:*
Three-Dimensional Solutions for the Free Vibrations and Buckling of Thermally Stressed Multilayered Angle-Ply Composite Plates, Journal of Applied Mechanics 59, 868-877 (1992)

[22] *Noor, A.K.; Kim, Y.H.; Peters, J.M.:*
Transverse Shear Stresses and Their Sensitivity Coefficients in Multilayered Composite Panels, AIAA Journal 32, 1259-1269 (1994)

[23] *Savoia, M.; Reddy, J.N.:*
Three-Dimensional Thermal Analysis of Laminated Composite Plates, International Journal Solids and Structures 32, 593-608 (1995)

[24] *Rolfes, R.; Noor, A.K.; Sparr, H.:*
Evaluation of Transverse Thermal Stresses in Composite Plates Based on First-Order Shear Deformation Theory, ASME Winter Annual Meeting, November 16-21, Dallas, Texas (1997)