

Three Dimensional Shape Optimization with Probabilistic Constraints using Parametric Session Files

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Probabilistic analysis is rapidly developing into a desirable tool to improve design processes. Incorporating probabilistics with optimization of three dimensional components is a step towards improving many current deterministic design systems. This paper develops a general purpose method using MSC/PATRAN and MSC/NASTRAN for three dimensional shape optimization that incorporates probabilistic calculations. A parametric finite element model calculates design point responses and semi-analytic geometry sensitivities. The Advanced Mean Value First Order Second Moment Method is used for reliability calculations while sensitivities to the probability constraints are calculated analytically. Demonstration problems are conducted on a cantilever beam, turbine engine disk, and turbine engine blade

INTRODUCTION

As structures become more advanced, requirements more ambitious, and operating environments more uncertain, it is increasingly important to rigorously account for variability in the design process. This variability is effectively accounted for with probabilistic methods. Efficient design requires combining these methods with optimization techniques to allow reliability-based optimization, a technique that potentially solves maintenance and cost goals of many organizations. Reliability-based optimization has been described in many papers but has been limited to application with beam and plate analysis, restricting the techniques to plate-like and axisymmetric structures.¹⁻⁶ Such models limit the flexibility of the technique and are unable to handle some issues key to the turbine engine community such as turbine disk bolt holes, and blade dovetails. A more general purpose capability would use three dimensional models.

Three dimensional shape optimization in commercial software commonly uses either parametric geometry models or definition of shape vectors. Sensitivity calculations with parametric geometry models frequently use either inefficient finite difference methods or potentially inaccurate response surface approaches. Technical publications have been presented on alternate methods for implementing parametric geometry optimization that are limited to two dimensions and not commercially supported as an effective design tool.^{7,8} An alternative to these parametric geometry models are shape vectors methods that allow efficient solutions and are integrated into MSC/NASTRAN.⁹⁻¹¹ Shape vectors describe the allowable finite element model shape changes and simplify geometric sensitivity calculation. Unfortunately, creating shape basis vectors can be technically challenging for complex geometry when capturing specific design intents. Therefore, a method that combines the ease of parametric geometry with the efficiency of basis vectors would provide a useful tool for designers interested in using probabilistic methods.

This paper develops such a methodology for three dimensional reliability-based optimization using parametric geometry. It is implemented with MSC/PATRAN, MSC/NASTRAN, DOT, and a FORTRAN program, REALITY, that integrates these products and adds probabilistic capabilities. The Advanced Mean Value First Order Second Moment Method is used for probabilistic calculations. Nodal sensitivities are generated with the parametric model to use for basis vector generation or analytic sensitivity analysis. Case studies of a cantilever beam, turbine engine disk, and turbine engine blade are conducted.

RELIABILITY OPTIMIZATION

When statistical variations of analysis input cause failure, a region of the system's joint probability density function (pdf) is intersected by a surface defining the transition from safe to failed designs. This surface is known as the limit state surface. The objective of probabilistic analysis is the calculation of the volume of the joint pdf that lies in the failure region defined by the limit state. The limit state equation is:

$$g(L, S) = L - S \quad (1)$$

where L is a limit on response defined for each problem (as a magnitude) and S is the response (such as a stress or displacement) calculated with a random variable vector. A *random variable vector* is the mathematical organization of design parameters defined by mean and standard deviation that is perturbed during probabilistics. A second vector of importance is the *design vector* which is the

mathematical organization of the design parameters to be perturbed during optimization. Each successive vector is ideally either closer to defining the probability of failure or determining the optimum design. Approximation techniques are required to efficiently calculate the probability of failure. One technique, and the method used for the case studies, relies on mean value methods.¹²⁻¹⁶ They require locating the limit state surface and approximating its curvature. The surface is found by defining the Most Probable Point, β :

$$\mathbf{b} = \frac{g(U^*) - \dot{\mathbf{a}} \sum_{i=1}^n \frac{\nabla g(U^*)}{\nabla R(i)} \mathbf{s}_{R(i)} u_i^*}{\sqrt{\dot{\mathbf{a}} \sum_{i=1}^n \left(\frac{\nabla g(U^*)}{\nabla R(i)} \mathbf{s}_{R(i)} \right)^2}} \quad (2)$$

where the limit state value and its sensitivity to the random variable vector, R , are required. In this equation U^* is the random variable vector transformed to a standard normal space and \mathbf{s} is the standard deviation. Once this point is found, the curvature of the surface is approximated. A common approximation is the First Order Reliability Method which uses a first order approximation. From eq. 2, it is important to note that probabilistic analysis requires the limit state value at a random variable vector, as well as the sensitivity of the limit state with respect to random variable vector.

Optimization techniques improve designs by using mathematical search algorithms to modify a design vector while maintaining constraints and minimizing objective functions. Like the probability calculations, the solution process is often one of linearizing nonlinear equations to generate an approximate solution, updating the design vector, then iterating until achieving convergence. Optimizing complex components is possible when search algorithms can manipulate a finite element model defined by design variables. In cases where three dimensional features are of interest, such as turbine disk bolt holes or turbine blade dovetails, use of complex shape variables are required. The fundamental requirements from the process are the value of the objective function (such as weight), constraints (such as stress/displacement limits), and the sensitivities of these values with respect to the design vector.

From the preceding two paragraphs, it is evident that probabilistic and optimization are dependent on sensitivity calculations. Because of the potential cost of such calculations it is critical to use efficient methods. Finite difference and response surface techniques are cost prohibitive when many random and design variables are present. Instead, analytic techniques are used to improve the performance of limit state and probabilistic objective function sensitivity calculations. Calculating limit state sensitivities with respect to the random variable vector, R , analytically enables gradient calculations at a much earlier and more efficient stage in the finite element formulation:

$$[K] \frac{\nabla \{u\}}{\nabla R} = \frac{\nabla \{P\}}{\nabla R} - \frac{\nabla [K]}{\nabla R} \{u\} \quad (3)$$

where the $[K]$ and $\{u\}$ are known. The sensitivities can be calculated by determining the variation of the stiffness matrix with respect to random variables. These sensitivities are identical to the sensitivity of the limit state equation with respect to changes in the random variables.

Reliability optimization also requires an efficient method for calculating probabilistic objective and constraint function sensitivities with respect to design variables. The variation of safety index with respect to the design variables vector is:

$$\frac{\nabla \mathbf{b}}{\nabla X} = \frac{1}{\sqrt{\dot{\mathbf{a}} \sum_{i=1}^n a_i^2 \mathbf{s}_{2i}}} \frac{\nabla g \mathbf{b}_X \mathbf{g}}{\nabla R} \quad (4)$$

where a_i is the sensitivity of the limit state to the i^{th} variable. Solutions of these equations, in conjunction with an optimization algorithm, allow for efficient probabilistic optimization. Implementation of these method with software is described in the next section.

TECHNICAL APPROACH

Three dimensional shape optimization with probabilistic constraints is demonstrated by integrating MSC/PATRAN, MSC/NASTRAN, and DOT with a FORTRAN program called REALITY. REALITY is a simple code that handles problem definition, search-and-replace capability, and file and execution control of other analysis codes. It also calculates grid location and safety index sensitivities in addition to conducting the probabilistic analysis. The proceeding paragraphs will describe and give examples of how the procedure is implemented in an automated fashion.

The fundamental requirement for conducting the optimization is the use of parametric geometry and its associated finite element entities. This can be done through the MSC/PATRAN session file that records commands executed during model construction and includes all geometry, material, load, and boundary condition information with numerical values. Also included in the file are the commands to output an MSC/NASTRAN Bulk Data File. The following is an example of a session file command that builds a rectangular solid by defining a point at the origin and a vector 20 in., 1.5 in., and 1.0 in. in the x , y , and z direction:

```
STRING asm_create_hpat_xyz_created_ids[VIRTUAL]
asm_const_hpat_xyz( "1", "<20.0 1.5 1.0>", "[0 0 0]", "Coord 0", @
asm_create_hpat_xyz_created_ids )
```

To make the command parametric, the numerical values are substituted with descriptive words:

```
STRING asm_create_hpat_xyz_created_ids[VIRTUAL]
asm_const_hpat_xyz( "1", "<length height width>", "[0 0 0]", "Coord 0", @
asm_create_hpat_xyz_created_ids )
```

Once this has been done to all necessary commands, the entire session file is parameter based. Of course, MSC/PATRAN cannot use this file as is because it has words where it expects numbers. The values of the current design or random variable vector need to replace the parameters. This is possible with a simple shell script designed to search the parameter based session file and replaces the words with their appropriate value. As an example, the shell program below defines the file to be modified, the parameter, and the numerical values that it will replace.

```
file=example.ses
word1=length
word2='20'
cat ${file} | sed -e "s/${word1}/${word2}/" >> junk.tmp
mv junk.tmp ${file}
```

Once the design point numerical data is entered into the session file, MSC/PATRAN is used to execute the session file in background mode with the following command line

```
patran -b -sfp example.ses
```

Running MSC/PATRAN in the background pauses REALITY during the creation of the MSC/NASTRAN input file. This file is then executed by MSC/NASTRAN to generate system responses and sensitivities (more information on sensitivity generation follows). Repeating this process allows solution of multiple finite element input files. While creating multiple files, it is critically important to control the number and location of the nodes and elements.

The parametric modeling technique must allow for two requirements. First, all geometry parameters must be defined with appropriate references as their definitions have large effects on perturbed geometry. Incorrect parameter definition can cause the technique to ignore a prescribed design intent. Second, numerically consistent finite element generation is required. In other words, for each design point iteration, the quantity and relative locations of nodes and elements must remain fixed. Such consistency can be facilitated by defining element quantities on an edge-by-edge basis through mesh seeding. The requirement for mesh consistency is problematic for tetrahedral mesh generators because of their somewhat unpredictable nature.

After the parametric session file is given the design or random variable vector information and the MSC/NASTRAN input file is generated and solved, results must be retrieved. A Direct Matrix Abstraction Program, DMAP, that dumps the results to a FORTRAN readable file is included in the Case Control section of the MSC/NASTRAN input file. Below is the format for the DMAP if both element stresses and nodal displacements are needed:

```
$ Direct Text Input for Executive Control
COMPILE SEDRCVR,SOUIN=MSCSOU,NOLIST,NOREF
ALTER 'OFP OES1X1,,,,,' $
type parm,,i,n,linespp $
linespp=getsys(linespp,9) $
paraml kelm/'trailer'/1/s,n,noelm $
noelm=20*noelm $
putsys(noelm,9) $
putsys(13,2) $
ofp oes1x1,ougv1 // s,n,cardno $
putsys(linespp,9) $
putsys(6,2) $
CEND
```

The output from the solver is parsed to retrieve the appropriate result quantity to be used for calculating eq. 2, the limit state equation. This process generates the response quantities at different design or random variable vectors that are required for the optimization and probabilistics calculation. This achieves only half the requirement, it still needs to be explained how to efficiently calculate sensitivities of the limit state and constraints to design and random variable perturbations.

When using MSC/NASTRAN SOL 200, response sensitivities for limit state sensitivities can be obtained from the grid coordinate sensitivities. These grid coordinate sensitivities can be calculated

by sequentially perturbing a geometric parameter, storing the change of nodal coordinates, and using finite difference calculations to give the sensitivity. This process is shown in Figure 1. These nodal coordinate sensitivities can then be given directly to MSC/NASTRAN through the DVGRID bulk data card. MSC/NASTRAN can then handle the math required to generate the sensitivities semi-analytically. The DVGRID results can be included in the bulk data file with the INCLUDE input card.

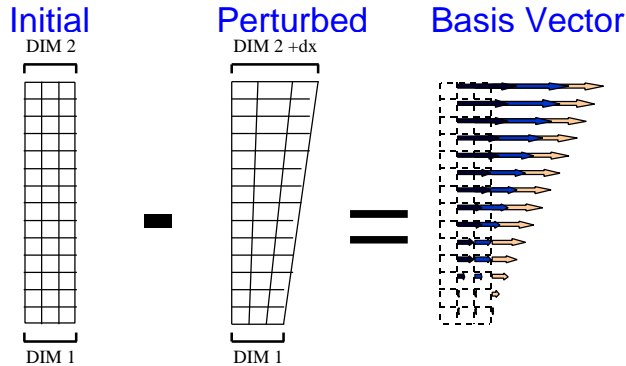


Figure 1: Geometric Basis Vector Generation Example

Solution of the bulk data file, with the included DVGRID cards, using SOL200, outputs the response, S , needed to solve the limit state equation (eq. 1). It also outputs the sensitivity of the S to changes in the random variable vector needed for eq. 2. These gradients are imported into REALITY and used for the probabilistic analysis. These same limit state gradients are used in REALITY to calculate the sensitivity of the safety index with respect to the design variable vector. The safety index is used as an optimization constraint in the optimization problem. The sensitivity of the objective function can also use these sensitivities if it is a function of the safety index. If it is a function of weight. MSC/PATRAN can be used to output the volume of the models it generates. These volumes can be retrieved through REALITY and finite difference methods used to calculate volume sensitivities to changes in the design vector.

All the calculation for an methods for efficient optimization with reliability constraints have been described. The process is iterative and automated in nature. First, design parameters are defined with the required mean and standard deviations. REALITY executes a probabilistic analysis loop that solves for a safety index. During the each probabilistic iteration, REALITY executes a second loop that calculates the sensitivities of the response to random variable vector changes. After convergence of the probabilistic analysis, the objective function and safety index constraint sensitivities are sent to an optimization routine. The design vector is updated given the information it receives about values and sensitivities, and begins the probabilistic loop again. This process continues until convergence. This process is shown in Figure 2.

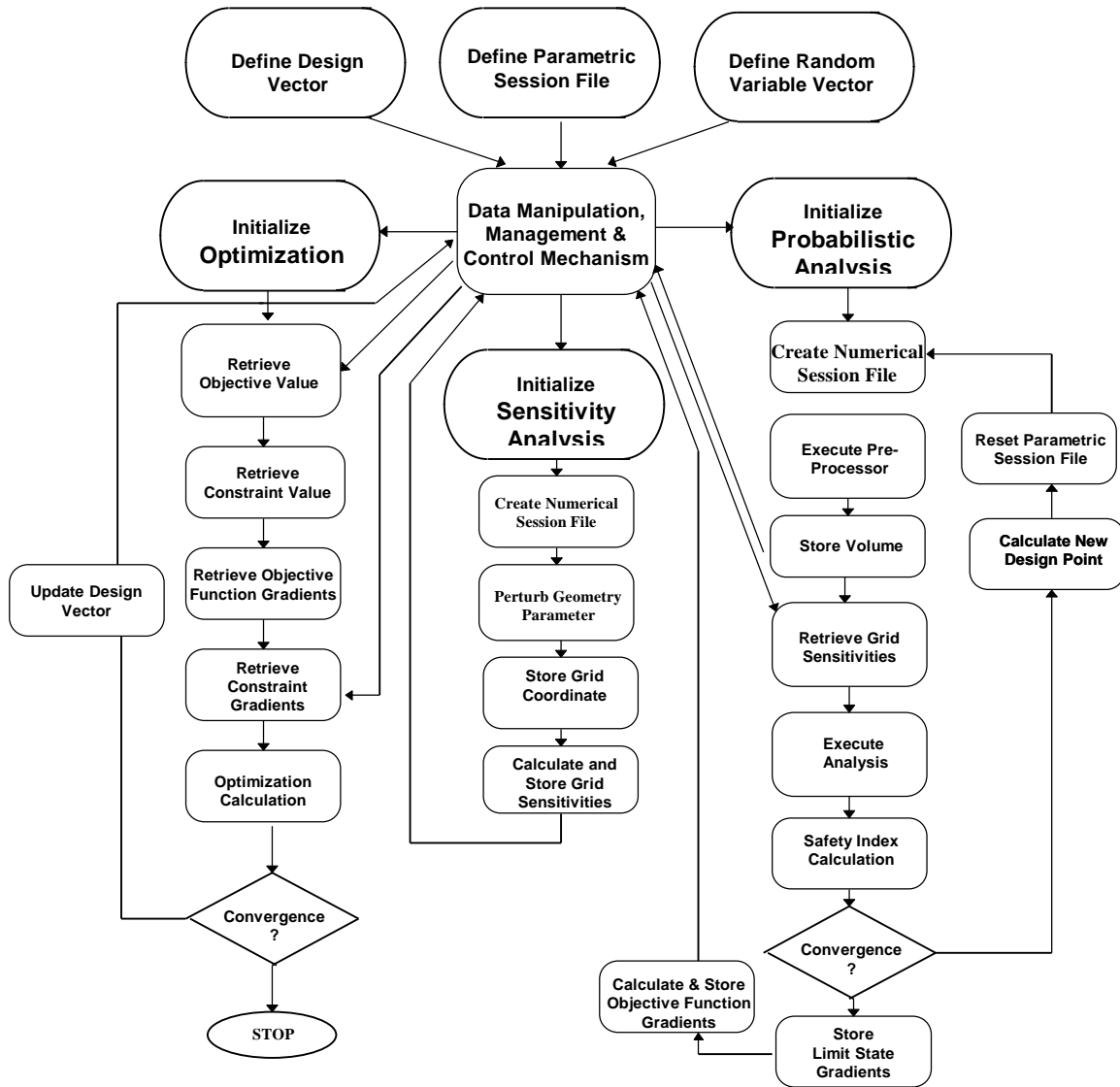


Figure 2: Reliability Optimization Methodology Flowchart

CASE STUDY

Three case studies demonstrate reliability-based optimization techniques on three-dimensional models. The structures selected were a cantilever beam, turbine disk, and turbine blade. The cantilever is a simple reference problem that can be compared with competitive techniques that exist or will be developed. The turbine disk example is important because of current efforts within the Air Force to incorporate probabilistics into the Engine Structural Integrity Program (ENSIP). Similarly, the blade example is relevant to the technical effort associated with the High Cycle Fatigue (HCF) Program. But first, a comparison of a parametric optimization approach versus an analytic boundary shape method available within MSC/NASTRAN is made.

Analytic Boundary Shapes vs. Parametric Method

A cantilever beam test case retrieved from the Test Problem Library was used to validate the parametric method in three dimensions. The beam is fixed at its base with a distributed transverse load applied to the tip. The optimization objective is to minimize volume while maintaining stress bellow allowable levels. In the MSC/NASTRAN test case, a pair of two dimensional auxiliary boundary models are created on the top and bottom surfaces of the beam with transverse loads applied at their tips to create cubic basis vectors. These basis vectors are controlled by two design variables that taper the height of the beam subject to a cubic function. Along with these auxiliary models, the nodes on the free faces are assigned BNDGRID bulk data cards.

In comparison, the parametric modeling technique defined the beam with one parameter, the beam height at the end of the beam. The beam was made parametric by assigning a parametric definition to the coordinate guiding beam height at the end of the beam. The geometry’s origin is located along the centerline of the beam so that the height of each section can be defined with a single value. So that only one design variable is needed, the height of the beam in the middle was linked to end design variable. A cubic interpolation was used to relate these two points so a third order curve can be made with a B-spline. The optimization results of the two solutions are provided.

Table 1: Comparison of Two Methods for Cantilever Beam Example

| Design Cycle | Objective (Analytic/Parametric) | Constraint (Analytic/Parametric) |
|--------------|------------------------------------|-------------------------------------|
| Initial | 80.00/80.00 | -0.038278/-0.038278 |
| 1 | 74.01/ 73.97 | -0.372091/-0.368142 |
| 2 | 65.63/ 65.50 | -0.357034/-0.346768 |
| 3 | 53.89/ 53.82 | -0.046728/0.013373 |
| 4 | 53.50/ 53.89 | -0.000808/-0.000140 |
| 5 | 53.50/ 53.95 | -0.000808/-0.000140 |

This set of results show that calculating grid sensitivities with the parametric model works as well as the Analytic Boundary Shapes method. In this case, an identical number of iterations were needed by both methods but the parametric method had a lower constraint value. This could be because of the interpolation used in the Analytic Boundary Shapes method.

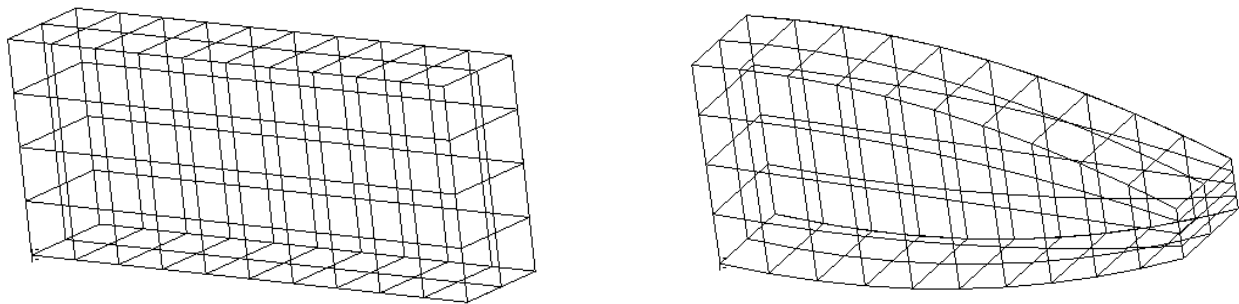


Figure 3: Initial and Optimized Cantilever Shape
Tapered Cantilever Beam

The initial cantilever beam had a 20" length, 1.5" width, and 1.5" height. The objective of the problem is to adjust the height of the beam along the length to minimize the volume and maintain a minimum safety index of 3.0, corresponding to a 0.001 probability of failure. Figure 4 presents the parametric model annotated with the three design variables (DIM1, DIM2, DIM3). A spline is constructed through these dimensions to define the entire boundary of the beam. For the probabilistic problem, a total of seven random variables were defined to calculate the safety index: the three design variables, beam length, Young's modulus, density, and transverse tip load. The mean values of the geometry parameters are identical to the geometry definition at each design iteration and the means of the modulus, density, and tip load are 30 Mpsi, 0.00044 slug/in³, and 1000 lb. respectively. All random variables were normally distributed with ten percent coefficients of variation, chosen because it increases the non-linearity of the probabilistic problem. Finally, for the limit state definition, a displacement of 0.75" was used as a failure limit, and the system response was monitored at tip.

Solution of the beam optimization with reliability constraints required a total of 63 MSC/NASTRAN calls. The initial safety index and volume were 0.628 and 30 in³ respectively. This safety index shows that the original structure is under-designed for the applied safety index constraint. The optimal design after 9 iterations had a safety index of 3.001 and volume of 31.42 in³.

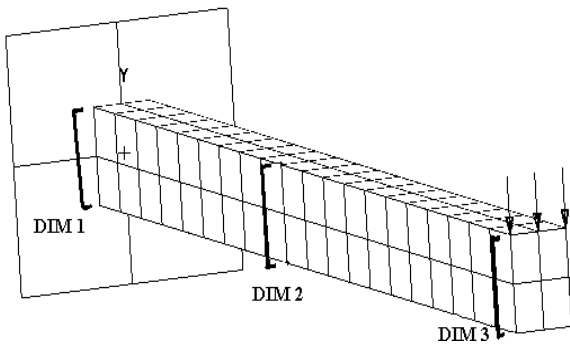


Figure 4: Parametric Cantilever Model

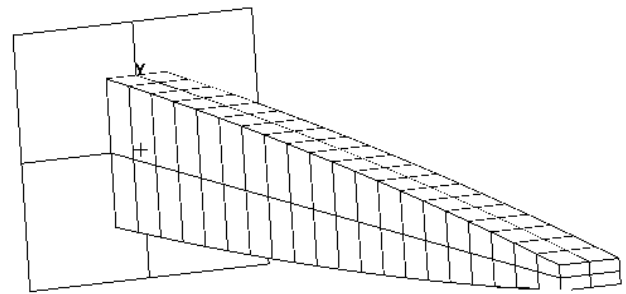


Figure 5: Optimized Cantilever Shape

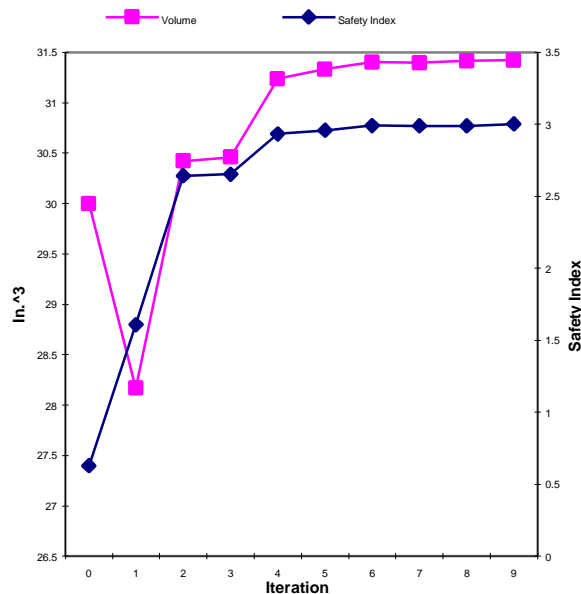


Figure 6: Objective and Constraint History

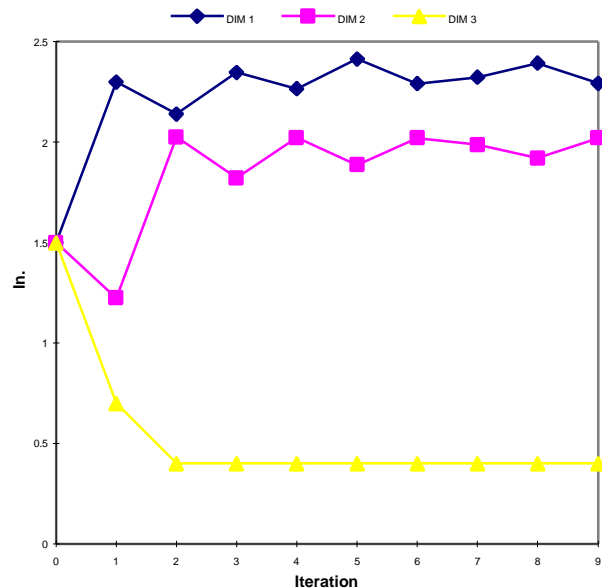


Figure 7: Design Variable History

The optimized shape is presented in Figure 5. Histories of the design variables, objective function, and constraints are shown in Figure 6 and Figure 7.

Turbine Engine Disk

A turbine engine disk, representative of actual engine hardware, is optimized. The inside and outside radius of the disk are at 1" and 6" respectively. The initial thickness of the bore is 1.5" and extends radially outward another 1". Beginning at the end of this section, the disk tapers to a minimum web thickness of 0.4". The thickness increases after this minimum. The rim of the disk is 1.5" wide so that it can accommodate an airfoil dovetail. To add a three dimensional feature to the geometry, a bolt hole was included at a 4" radius. This increased the modeling complexity and showed the robustness of parametric modeling capabilities. Figure 8 displays the parametric model with annotated locations of the design variables.

The volume of the disk was used as the objective function. Obviously, reducing this volume has a direct relationship to weight reduction, a primary goal for turbine engine hardware. For the reliability constraint, a minimum safety index of 3.5, corresponding to a 0.0002 probability of failure was used. Additional deterministic constraints are placed on the design variables to prevent unacceptable geometry changes. The lower limit on the design variables was 0.4".

The only values left to define are for the probabilistic analysis. Eight random variables were used: the five geometry design variables, modulus, density, and rotational velocity. The mean values of the geometry parameters are identical to the initial geometry definitions at each design point, and the means of the modulus, density, and rotational velocity are 30 Mpsi, 0.00075 slug/in³, and 285 rev/sec, respectively. All random variables were normally distributed with ten percent coefficients of variation. It is important to understand that these statistical parameters are used only for demonstration purposes because there was not an available source of actual variations. A ten percent variation is most likely the upper limit of variability for the selected parameters. Later in this section, a parametric study on the variation effects of the statistical distributions on the results is presented.

Definition of the limit state for this study required more effort. Turbine disk optimization ideally requires a system reliability approach to define the failure surface because it is important to calculate failure probability at multiple locations. For example, in deterministic optimization, the stress at every element is constrained to certain stress limits. This is critical because shape modifications effect the global response of the structure. For this study, the reliability calculations were made at the bore, bolt hole, and at the top of the disk web. These locations cover the entire length of the disk and prevent regions of the structure from being ignored. For this problem, a failure limit was defined at a 75 ksi Von-Misses stress.

The finite element model employed techniques to reduce computational cost while maintaining accuracy. First, a forty-five degree segment was modeled with

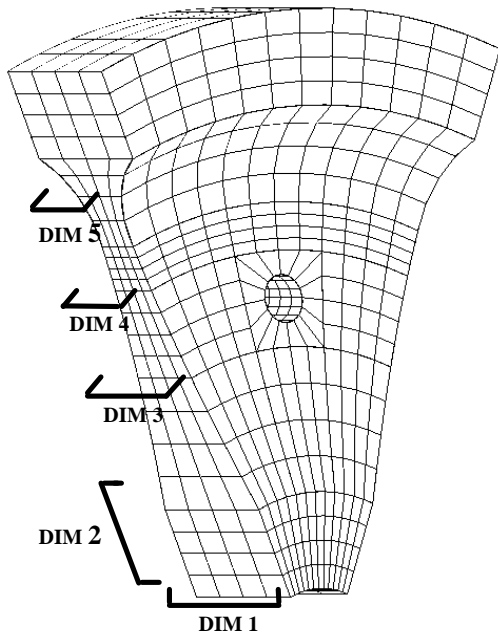


Figure 8: Parametric Turbine Disk Model

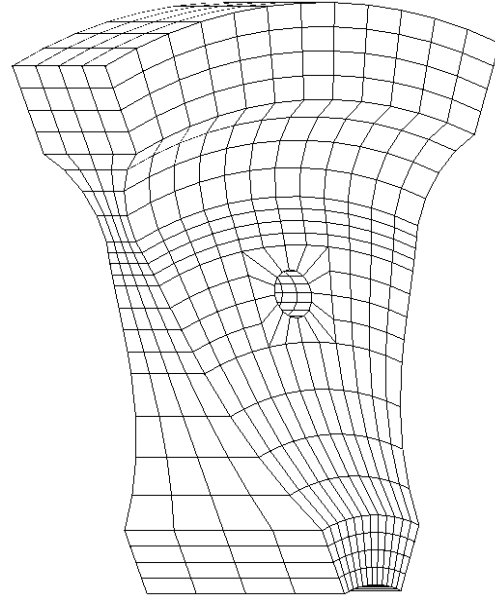


Figure 9: Optimized Turbine Disk Model

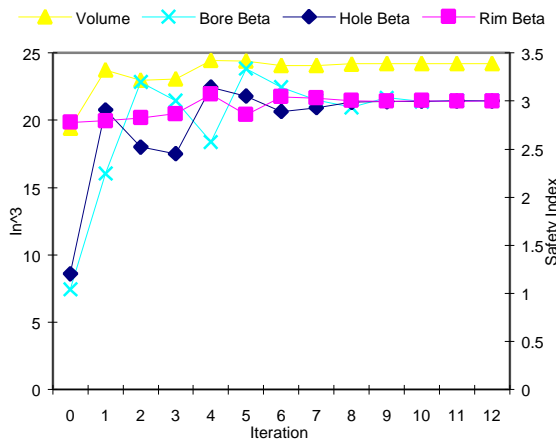


Figure 10: Disk Optimization History

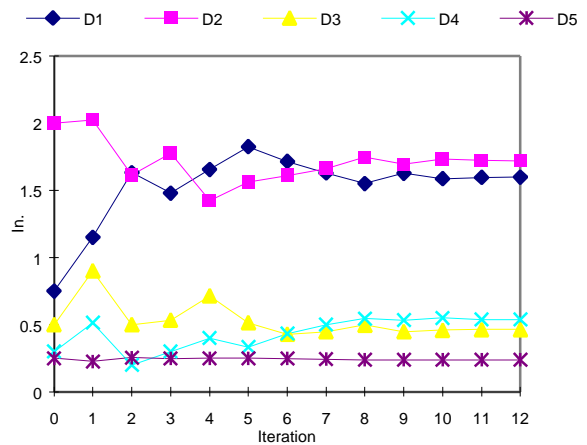


Figure 11: Disk Variable History

appropriate symmetry constraints to ensure that the disk was rotationally symmetric. This reduces the physical size of the model by almost 90 percent. Also, a relatively coarse mesh was used to further reduce the computation times. The coarse mesh will not influence stress predictions at the bore and web significantly. It will most influence the stress calculations at the bolt hole, but the effect should be relatively minor. To further minimize complexity, it is desirable to account for only the most significant loading and boundary conditions. This will reduce the number of random parameters in the problem. For this case study, only rotational loading was applied to the model. Also, the bore of the rotor was fixed in the axial direction, along the direction of the shaft. Other forces that affect the stress such as blade mass, aerodynamic forces, temperatures, and external structural constraints are not included. They were neglected because they would only affect the magnitude of stress and would

not add to the difficulty of the solution. Since this problem is a demonstration, the rigorous inclusion of loads and boundary conditions was neglected.

Using this parametric model the probabilistic problem was successfully solved and the optimal configuration is shown in Figure 9.

For the turbine engine disk problem, the optimization required 317 MSC/NASTRAN solutions. The initial safety index and volume was 1.04 at the bore, 1.21 at the bolt hole, 2.77 at the minimum web thickness, and 19.45 in³, respectively. Again, this shows that the initial structure was under-designed for the defined safety index constraint. The optimal design after 9 iterations met the 3.5 safety index constraint and increased the volume to 24.216 in³. The histories of this process are shown in Figures 10 and 11.

Turbine Engine Blade

The airfoil represents a typical turbine blade and is approximately 6" long, 3" wide, and has about 30 degrees of twist. The maximum blade thickness is 0.28" at the fillet with a minimum thickness of 0.03" at the leading edges and tip. The root and platform geometry were not included in the finite element model. For the optimization problem, a minimum safety index of 3.5, corresponding to a 0.0002 probability of failure, was required while minimizing the volume. The thickness of the blade at 15 locations, shown in Figure 12, were used as design variables. The lower and upper limit set on the design variables were 0.03" and 0.35". Further geometry constraints were applied that enforced airfoil like geometry, keeping the curvature of cross sections from inverting and forcing the thickness to decrease in the radial direction.

The reliability analysis needed for the constraint calculation consisted of a total of 18 random variables: 15 geometry design variables, modulus, density, and rotational velocity. The mean values of the geometry parameters are identical to the initial geometry definitions at each design iteration, and the means for the modulus, density, and rotational velocity are 16 Mpsi, 0.0004 slug/in³, and 125 rev/sec, respectively. All random variables were normally distributed with ten percent coefficients of variation. The limit state for this problem used deflection of a node at the blade to create the failure surface. The system response was calculated at the leading edge of the blade tip, and the failure limit was defined at a 0.125 deflection. This type of limit state was used because studies are currently being conducted of relationships between tip deflection and stress for use with Non-intrusive Stress Measurement Systems, NSMS. Therefore, in the near future, such a tip deflection could be used as a failure indicator.

As with the turbine disk test case, only the most significant loads and boundary conditions were applied. Rotational loading was applied to the model with the base of the blade fixed in all degrees of freedom. Other forces that contribute to the blade response include aerodynamic forcing and temperature but were not included in the model. Typically these forces will increase stresses by only 10 to 20 percent for a fan blade. Addition of these loads would not greatly affect the outcome of the optimization and were neglected.

The original volume and safety index of the blade were 2.58 in³ and 1.56. Again, the original structure is under-designed for the safety index constraints. The solution converged to the optimal design in 12 iterations with a volume of 3.18 in³. Figure 13 shows the three dimensional geometry of the optimal configuration. Inspection of the optimized geometry shows that the thickness increased in the bottom and middle airfoil section and reduction in area in the tip cross section. Figure 14 shows the history of the optimization.

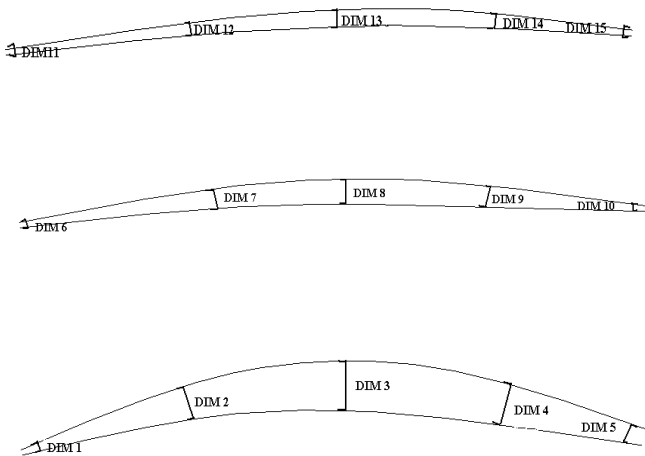


Figure 12: Parametric Airfoil Model

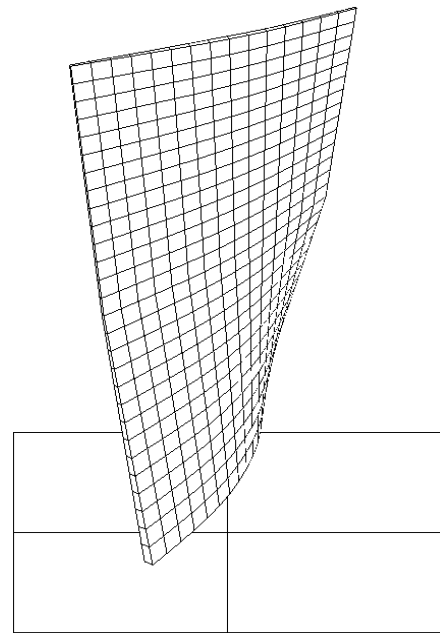


Figure 13: Optimum Airfoil Geometry

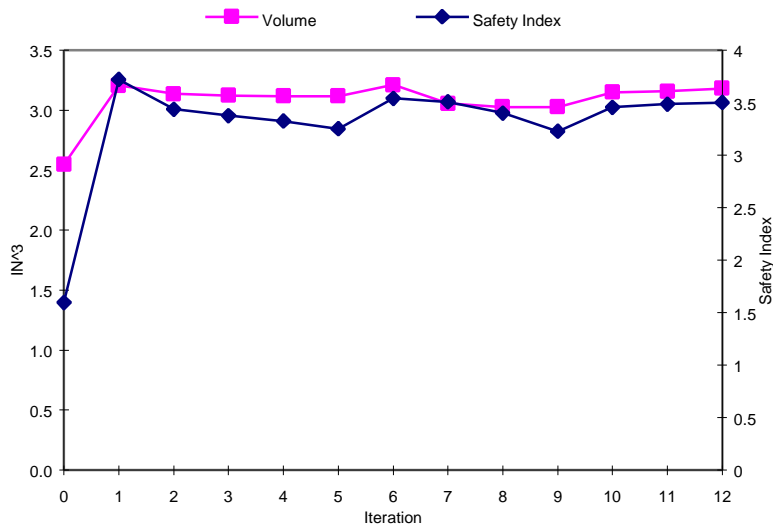


Figure 14: Blade Optimization History

Summary and Conclusions

Reliability optimization is a design tool that allows engineers to gain greater insight into complex engineering problems that involve statistical variations. A majority of the research into this technology, and optimization in general, has been limited to two dimensional finite element models because of the straightforwardness of design variable input. Such two dimensional problems are effective for plate-like and axi-symmetric structures, but they lack the capability to model a three dimensional feature such as a turbine engine disk with bolt holes. This paper presented a technique to conduct reliability optimization on three dimensional components using a parametric finite element

model. It efficiently calculates sensitivities by using MSC/NASTRAN's semi-analytic sensitivity capabilities.

The parametric techniques described in this paper are applicable to a wide array of situations. It can be used for any process that requires multiple iterations with design perturbations. Other work conducted by the Air Force Research Lab has used these techniques. One involved a rotor mistuning system that investigates system response changes from small blade geometry changes. Another involves structural tailoring of engine blades to adapt to optimized aerodynamic shapes. The technique also allows itself to be integrated with a variety of other software. This is what should be of primary interest to most MSC users.

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