Modal Analysis - An Elegant Solution

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Abstract

In FEM analyses for large and complex structures small structural elements are mostly modelled by scalar elements. These scalar elements represent either stiffness or mass properties. In a linear bifurcation analysis, not only the stiffness matrix but also the geometrical stiffness matrix is needed. However, scalar stiffness elements, as well as GENEL elements and DMIG stiffness input do not possess such geometrical stiffness matrices. Therefore these elements obstruct the buckling analysis. To overcome this problem, a non-linear analysis in conjunction with a classical normal mode analysis is proposed. Zero eigenvalues will define the buckling load and associated buckling mode. The approach will be discussed in this paper and will be illustrated with the buckling analysis of an in-plane loaded simplified deployed solar array.

Keywords: Spacecraft Structures, Solar Arrays, Buckling, Non-linear Static Analysis, Normal Mode Analysis.

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Introduction

Fokker Space investigates the feasibility of a High Power Solar Array (HPSA) for a TBD mission. The major requirements for the HPSA are: power in KW Begin of Life (BOL) and End of Life (EOL), the operating voltage (DC), the power to weight ratio, stiffness requirements (natural frequencies) in stowed and deployed position, launch and test loads and the on orbit loads, in particular reboost loads on the fully deployed wing. The reboost occurs when the spacecraft is manoeuvring from one orbit to another orbit. It must be proved that the deployed wing shall survive the reboost loads. The analysis which was carried out for this was the motive for this paper. Buckling is considered to be the critical failure mode. Therefore the allowable buckling load had to be calculated with the available MSC/NASTRAN [Kilroy] FEM of the deployed wing.

It is common practice to begin with the calculation of the linear lowest bifurcation point (load) using solution sequence 105 (SEBUCKL). The results of this analysis showed many spurious buckling modes, more or less “rigid body” modes. The calculated eigenvalues were very low figures without any physical meaning . The explanation of this phenomenon is given below.

When the HPSA wing is deployed, the yoke, the solar panels and the PEF’s are connected to each other by means of locked hinges. In general these locked hinges are idealised with simple scalar and mass elements; respectively CELASi elements to represent the stiffness and CONM2 elements to represent the mass. The SADM stiffness as well is represented by scalar spring elements.

The presence of scalar spring elements obstructs the calculation of the linear bifurcation point (SOL 105). Scalar spring elements don’t contribute to the geometrical stiffness matrix. Consequently there is no coupling between the stiffness matrices of yoke and panels in these geometrical stiffness matrices. This will result in very low numbers of the eigenvalues.

[Lee] discussed the non-linear buckling analysis. However at a certain level of the applied load the buckling analysis is linearized. In the linearised equilibrium state the buckling analysis is analogue to the linear bifurcation analysis. Again a geometrical stiffness matrix must be generated. Again there is no contribution of the scalar spring elements.

Discussions with [Riks] led to the following idea (procedure) to calculate the allowable buckling load with a FEM which containing, amongst other elements, scalar spring elements:

1. The bifurcation point or limit point of a loaded structural system is located with a standard Geometrical Non-Linearity (GNL) analysis, [Becker].
2. After the GNL analysis the corresponding buckling mode at the bifurcation point or limit point is calculated with a standard normal mode analysis using adequate parameter setting (NMLOOP), [Lee].

This paper contains the following chapters; the *problem definition* in which the problem will be defined in more detail, *analysis* where the basic theory concerning stability will be briefly recalled and demonstrated by means of two simple example problems which are solved theoretically and with the aid of MSC/NASTRAN, a *discussion* of the proposed method and finally the *conclusions* and *recommendations* are given.

The proposed method to calculate the bifurcation or limit point and associated buckling modes is used for a typical deployed HPSA loaded with in plane inertia loads.

Problem Definition

The linear buckling analysis method with MSC/NASTRAN is quite popular. The analysis, using solution sequence 105, is very straight forward. However, in general the calculated eigenvalues are optimistic and thus too high and must be regarded with a certain amount of engineering experience. Geometrical imperfections will lower the allowable buckling load. For thin-walled circular cylinders [NASA1] and truncated cones [NASA2] knockdown factors are applied to obtained more reliable allowable buckling loads (stresses).

The calculation of the linear bifurcation point is based on the eigenvalue problem constituted from the linear stiffness matrix $[K]$ and the geometrical stiffness matrix $[K_G]$ and is defined as:

$$\left([K]+\lambda_k [K_G]\right)\{u_k^c\}=\{0\} \qquad \textbf{eq. 1}$$

The critical load becomes: $\{F_{cr,k}\}=\lambda_k \{F\}$

A “rigid” solar array consists of a number of “rigid” sandwich panels covered with solar cells. The panels are interconnected with hinges. To keep the panels in deployed configuration out of the shadow of the spacecraft (S/C), the solar panels are connected to the S/C via a standoff, in general a yoke. The yoke is at one end hinged to the solar panels and at the other end to the SADM. In deployed position the hinges are locked in order to maintain a maximum natural frequency.

In general the FEM of the S/A consists of quadrilateral membrane and plate bending elements (CQUAD) and extensional and bending line elements (BAR, BEAM). In general the hinges are idealised with scalar spring elements (CELAS) to represent the locked hinge stiffness. Either this stiffness is measured or obtained from a detailed FEA.

The incorporation of scalar spring elements obstructs the linear bifurcation analysis (SOL 105). The scalar spring elements do not posses any geometrical stiffness capabilities, causing analysis problems. The solar panels and the yoke do have geometrical stiffnesses that depend upon the occurring stress (tension, compression) due to the applied loads. The geometrical stiffness matrix $[K_G]$ consists of separate stiffness matrices of the solar panels and the yoke. No connection is made by the scalar spring elements. As a consequence one can state that no reliable buckling analysis is possible when scalar spring elements are used in the FEM

It is investigated how the problems which are caused by the use of scalar elements can be solved for this kind of analyses. Allowable buckling loads must be predicted in order to calculate the margin of safety with respect to the occurring loads. The proposed analysis method to calculate the allowable buckling loads eliminates the problems caused by the scalar spring elements.

The procedure to calculate the buckling loads with a FEM containing scalar spring elements (there is no need to avoid the use of scalar spring elements) comprises two steps:

1. Calculate the tangent stiffness matrix $[K_t]$ either at the first appearing bifurcation point or at the limit point. The stiffness matrix $[K_t]$ may be obtained during a GNL analysis with for example the arc-length solution technique [Hinton], [Lee] (SOL 106). Passing bifurcation points or a limit point may be monitored with the number of times the determinant of the tangent stiffness matrix changes sign (so-called number of negative terms on main diagonal). However, the indication of passing a bifurcation point or a limit point is not straightforward. Only the lowest bifurcation point or the first limit point is of interest, meaning that the state of equilibrium changes from stable too unstable. This change marks the bifurcation point or buckling load, however, no information is available about the buckling mode.
   In the neighbourhood of the bifurcation point (approximated from below) a linearized bifurcation analysis (analogue to SOL 105) is done. The generation of the geometrical stiffness matrix will cause also problems with the scalar spring elements.

2. In this step of the procedure the buckling mode at a bifurcation point or limit point will be calculated. This is a standard dynamic eigenvalue problem $\left([K_t]-\tilde{\omega}_k [M]\right)\{\delta u_k^c\}=\{0\}$. In case of a lowest bifurcation point or limit point the eigenvalue $\tilde{\omega}_1=\{0\}$ and $\{\delta u_k^c\}$ $\left([K_t]\{\delta u_k^c\}=\{0\}\right)$is the buckling mode, corresponding with $\|K_t\|=0$.

In the next chapter the theory of the proposed approach will be described and illustrated with two simple stability problems; Ziegler’s two dof's cantilevered model [Wu] , the propped cantilever [Ben-Haim] to investigate the procedure in case of imperfections and a real life structural problem; the allowable buckling load of the HPSA exposed to in-plane inertia loads.

Analysis

Stability Analysis

When a structure is elastic and applied loads are conservative, an energy function or rather the potential energy Π of the structure-load system can be defined as $\Pi = U - W$. An equilibrium state of the system can be obtained using the principle of stationary potential energy:

$$\delta\Pi = 0 \qquad \textbf{eq. 2}$$

The stability analysis can be reduced to a test of positive definiteness (or existence of a strict minimum) of the total potential energy (TPE) of the structure-load system. The equilibrium state at $\{u + \delta u\}$ is stable for those values of the applied load $\{P\}$ for which:

$$\delta^2\Pi = \frac{1}{2!}\frac{\partial^2(\{u\}, P)}{\partial u_i \partial u_j}\delta u_i \delta u_j > 0 \text{ for any } \delta u_i \text{ and } \delta u_j \qquad \textbf{eq. 3}$$

The tangential stiffness matrix of the structure system is:

$$K_t(i, j) = \frac{1}{2}\frac{\partial^2(\{u\}, P)}{\partial u_i \partial u_j} \qquad \textbf{eq. 4}$$

In the critical state the second variation of the potential energy tends to zero $\{\delta u\}^T[K_t]\{\delta u\} = 0$ for some vector $\{\delta u\} = \{\delta u^c\}$. The critical state of stability may be detected from the condition:

$$\|K_t\| = 0 \qquad \textbf{eq. 5}$$

The calculation of $\|K_t\| = 0$ is equivalent to solving an eigenvalue problem:

$$([K_t] - \omega_k [I])\{\delta u_k^c\} = \{0\} \qquad \textbf{eq. 6}$$

All eigenvalues are positive for a stable equilibrium. One or more negative eigenvalues indicate unstable equilibrium states. A critical state will be indicated when the eigenvalue is zero with $\{\delta u_k^c\}$ as the buckling mode. Then the matrix equation:

$$[K_t]\{\delta u_k^c\} = \{0\} \qquad \textbf{eq. 7}$$

has a nonzero solution $\{\delta u_k\} = \{\delta u_k^c\}$ in the critical state.

In equation (6) the identity matrix can be replaced by the mass matrix $[M]$ without changing the buckling eigenvalue problem. The eigenvalue problem becomes:

$$([K_t] - \tilde{\omega}_k [M])\{\delta u_k^c\} = \{0\} \qquad \textbf{eq. 8}$$

A critical state (bifurcation or limit point) will be indicated when the eigenvalue $\tilde{\omega}_k = 0$ with $\{\delta u_k^c\}$ as the buckling mode. [Simitses] qualifies this as the “Kinetic or dynamic” approach. If the state of equilibrium is stable the eigenvalue must $\tilde{\omega}_k > 0$. In case of unstable equilibrium the eigenvalue must $\tilde{\omega}_k < 0$. The number of negative eigenvalues is an indication about the passed bifurcation points and or limit point.

Example 1: Ziegler's two dof's cantilevered model

A simple example has been chosen from [Wempner]. [Wu] calls this problem "Ziegler's two-degree-of-freedom model".

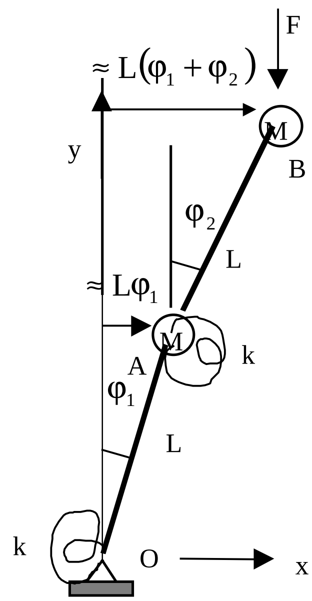


Figure 1 "Ziegler's two-degree-of-freedom model".

The system of Figure 1 consists of two rigid links with a length L, OA and AB, pinned at O and A. The relative motions at O and A are resisted by linear torsion springs with a spring stiffness k. The system is subjected to a constant force F in vertical direction. In the prebuckling state are both angles $\varphi_1 = \varphi_2 = 0$. The springs are undeformed and the links are aligned. At the points A and B discrete masses M are attached. These masses are used in the dynamic approach. The critical load F_{cr} will be calculated using the minimum potential energy Π approach and the dynamic approach $(\tilde{\omega} = 0)$.

Theoretical solution

Minimum Potential Energy

The total potential energy of the structure-load system is;

$$\Pi = U - W = \frac{1}{2}k\varphi_1^2 + \frac{1}{2}k(\varphi_2 - \varphi_1)^2 - FL\{(1 - \cos\varphi_1) + (1 - \cos\varphi_2)\} \qquad \textbf{eq. 9}$$

$\delta\Pi = 0$ gives with $\sin\varphi \approx \varphi,\ \varphi << 1$:

$$k\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = FL\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} \qquad \textbf{eq. 10}$$

The critical load is a result of the eigenvalue analysis:

$$F_{cr} = \frac{3k}{2L}\left(1 - \frac{1}{3}\sqrt{5}\right) = 0.38197\frac{k}{L} \qquad \textbf{eq. 11}$$

The buckling mode is: $\frac{\varphi_2}{\varphi_1} = \frac{FL - 2k}{k} = 1.62$

The second variation of the total potential energy is:

$$\delta^2 \Pi = (FL)^2 - 3k(FL) + 2k^2 \qquad \textbf{eq. 12}$$

$$\delta^2 \Pi = \begin{cases} < 0,\ F > 0.38197 \frac{k}{L} \rightarrow \text{unstable} \\ = 0,\ F = 0.38197 \frac{k}{L} \rightarrow \text{critical} \\ > 0,\ F < 0.38197 \frac{k}{L} \rightarrow \text{stable} \end{cases}$$

Dynamic Approach

The linearized undamped equations of motion of the dynamic system, with the generalised degrees of freedom φ_1 and φ_2, are:

$$ML^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{Bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{Bmatrix} + k \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = FL \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} \qquad \textbf{eq. 13}$$

The lowest eigenvalue is:

$$\tilde{\omega}_1 = \frac{1}{ML^2} \left[3k - \frac{3}{2} FL - \frac{1}{2} \sqrt{32k^2 - 24kFL + 5F^2L^2} \right] \qquad \textbf{eq. 14}$$

In case the applied load $F = 0$ the lowest eigenvalue becomes: $\tilde{\omega}_1 = \frac{k}{ML^2} \left[3 - \frac{1}{2}\sqrt{32} \right] = 0.17157 \frac{k}{ML^2}$

The eigenvalue becomes zero when $F_{cr} = F \rightarrow \frac{1}{2}\frac{k}{L}\left(3 - \sqrt{5}\right) = 0.38197 \frac{k}{L}$. The critical load is independent of the mass distribution. The corresponding buckling (vibration) mode is $\frac{\varphi_2}{\varphi_1} = \frac{2k - FL}{k} = 1.6180$

MSC/NASTRAN solution

The finite element model consists out of the two “rigid” bars, with a length $L = 1$ [m] connected with a rotational spring element (A) and “hinged” supported at the lower side with a rotational spring (O) . Both springs have a spring stiffness $k = 1$ [Nm/rad] about the z-axis. The finite element is positioned in the x-y plane. Both links are positioned on the y-axis ($\varphi_1 = \varphi_2 = 0$ [Rad]). A concentrated load of $F = 0.38197$ [N] has been applied at the top (B, y=2 [m]), pointing in the –y direction. Only the dof’s u, v and φ_z are considered. Both “rigid” links are modelled with 10 CBEAM elements with a bending stiffness $EI = 1.0e5$ $[Nm^2]$ and the mass per unit of length $m = 1$ [kg / m]. A lumped mass matrix had been generated. In case of a linear modal analysis a lowest eigenvalue of $\omega_1^2 = 0.34379$ is found. A linear bifurcation analysis (SOL 105) resulted in a bifurcation load $F_{bif} = 0.361128$ [N]. The errors of the linear bifurcation load (SOL 105) with respect to the theoretical bifurcation load is $\frac{F_{theor} - F_{bif}}{F_{theor}} * 100\% = \frac{0.38197 - 0.36113}{0.38197} * 100\% = 5.46\%$.

The concentrated load $F = 0.38197\ [N]$ has been applied with 10 increments using the Riks's method [Lee]. The static analysis is non linear and carried out with the MSC/NASTRAN (V70.5.0) solution sequence 106 (NLSTATIC). Once the maximum concentrated load has been reached the solution sequence is changed to a normal mode analysis using the parameter control (PARAM,NMLOOP,10) and the lowest natural frequency with the associated mode (buckling mode) is calculated.

The lowest calculated eigenvalue is $\tilde{\omega}_1 = -6.53\ 10^{-06}\ \left[(Rad/s)^2\right]$. The associated (buckling) mode is $\frac{\varphi_2}{\varphi_1} = 1.62$.

Example 2: Propped cantilever with imperfection Sensitivity

A pinned link with length L [m] has a specific mass $\tilde{m}\ [kg/m]$ and is supported by a linear spring with a spring stiffness $k\ [N/m]$ (tension and compression). The link is loaded with a constant force $F\ [N]$.

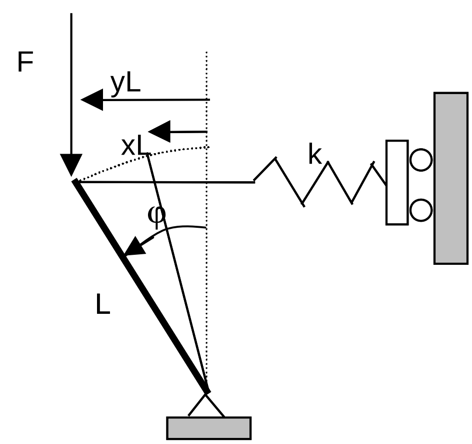


Figure 2 Propped Cantilever [Ben-Haim]

The initial imperfection is modelled with a deflection $xL\ [m]$ and the total deflection of the link is $yL\ [m]$.

The critical load F_{cr} occurs for $y = x^{\frac{1}{3}}$ [Ben-Haim] :

$$F_{cr} = kL\left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \qquad \textbf{eq. 15}$$

The following table gives the sensitivity of the critical load F_{cr} as function of the initial imperfection

x	F_{cr}
0.0	kL
0.1	0.6949kL
0.2	0.5338kL

The equation of motion with respect to the hinge of the rigid link is for the undamped motion given by

$$\frac{1}{3}\tilde{m}L^3\ddot{y} + kL^2\left\{(y-x)\sqrt{(1-y^2)}\right\} - FLy = 0 \qquad \textbf{eq. 16}$$

This equation of motion is non-linear. As a consequence the calculation of the "natural frequency" is not straightforward. This equation must be linearized around a state of equilibrium

$$kL^2\left\{(y-x)\sqrt{(1-y^2)}\right\} - FLy = 0 \qquad \textbf{eq. 17}$$

The linearized undamped equation of motion yields $\frac{1}{3}\tilde{m}L^3\ddot{y} + kL^2 y - FLy = 0$, which means that the critical load at x=0 equals $F_{cr} = kL$. The natural frequency around x=0 and k=1, L=1, $\tilde{m}$ =1 equals:

x	ω_k^2	f_k [Hz]
0.0	3	0.4775

MSC/NASTRAN solution

In the finite element model of the propped cantilever the "rigid" bar, with a length $L = 1$ [m] , is "simply supported" at the lower side and at the upper side supported with a linear spring. The spring has a spring stiffness $k = 1$ [N/m]. The finite element is positioned in the x-y plane. A concentrated load of F $[N]$ is applied at the top. The "rigid" link is modelled with 10 CBEAM elements with a bending stiffness of $EI = 1.0e5\ [Nm^2]$ and a specific mass of $m = 1\ [kg/m]$. A lumped mass matrix is generated. The rotation φ resulting from the applied load is calculated. The calculated bifurcation point and the two limit points are estimations:

x	F_{cr}
0	0.9999

x	3-order polynome F_{cr}	4-order polynome F_{cr}	$\tilde{\omega}_{kc}$ (3-order)	$\tilde{\omega}_{kc}$ (4-order)
0.1	$F_{cr} = 0.6960$	$F_{cr} = 0.6951$	1.266.7	0.1850
0.2	$F_{cr} = 0.5341$	$F_{cr} = 0.5339$	0.5672	0.2455

The natural frequency is very sensitive for the location of the equilibrium state near the limit point. If one assumes slow varying vibration modes in the neighbourhood of the limit point, the "buckling" mode is obtained even when the natural frequency $\tilde{\omega}_{kc} \neq 0$. It is only a thought and no proof will be given.

The indirect search for a bifurcation point

By following the equilibrium path, using the path-following method for solving the equations of equilibrium, one is able to pass a bifurcation point. Below this point the equilibrium state is in general stable and after that point unstable. At the bifurcation point several states of equilibrium do exist and the equilibrium state will branch to another path where the state of equilibrium remains stable. In case of structures with an imperfection, the bifurcation point will be manifest as limit point. By passing bifurcation points or a limit point, the determinant of the tangent stiffness matrix will change sign or $[K_t]\{\delta u_c\} = 0$. When the bifurcation or limit point is passed, MSC/NASTRAN issues a warning: "The number of negative term on the factor diagonal = n ". This message is a result of the decomposition of the tangent stiffness matrix. Therefor the stiffness matrix shall be updated for every step in the neighbourhood of the bifurcation point (KSTEP=1 in the NLPARM BDD record). It is very difficult to spot a bifurcation point exactly. The param, TESTNEG, n may be very helpful.

A critical state (bifurcation or limit point) is indicated when the eigenvalue $\tilde{\omega}_{kc} = 0$, with $\{\delta u_k^c\}$ as the buckling mode. If the state of equilibrium is stable the eigenvalue must $\tilde{\omega}_{ks} > 0$. In case of an unstable equilibrium the eigenvalue must $\tilde{\omega}_{ku} < 0$. The number of negative eigenvalues is an indication about the passed bifurcation points and or limit point.

The following solution procedure is proposed [Seydel]. With $\Delta\tilde{\omega}_k = \tilde{\omega}_{ku} - \tilde{\omega}_{ks} \le 0$, $\tilde{\omega}_{ku}\tilde{\omega}_{ks} \le 0$ and $\Delta F_k = F_{ku} - F_{ks}$, using the bisection technique, the first estimate of the bifurcation load F_{kc} at the critical state can be estimated with:

$$F_{kc} = F_{ks} - \tilde{\omega}_{ks}\frac{\Delta F_k}{\Delta\tilde{\omega}_k} \qquad \textbf{eq. 18}$$

The estimation of the eigenvalue at the critical state may be $\tilde{\omega}_{kc} = \tilde{\omega}_{ks} + \mu\Delta\tilde{\omega}_k$ with $\mu \ge 0$. We know that $\tilde{\omega}_{kc} = \tilde{\omega}_{ks} + \mu\Delta\tilde{\omega}_k = 0$, hence $\mu = \frac{-\tilde{\omega}_{ks}}{\Delta\tilde{\omega}_k}$ with $\tilde{\omega}_{ku}\tilde{\omega}_{ks} \le 0$ and $\Delta\tilde{\omega}_k$ typically small when bifurcation's are investigated. The critical load can be estimated as $F_{kc} = F_{ks} + \mu\Delta F_k$. The following example illustrates the indirect search for a bifurcation point.

$\tilde{\omega}_1 = \frac{1}{ML^2}\left[3k - \frac{3}{2}FL - \frac{1}{2}\sqrt{32k^2 - 24kFL + 5F^2L^2}\right]$, the eigenvalue becomes zero if :

$F \to \frac{1}{2}\frac{k}{L}\left(3 - \sqrt{5}\right) = 0.38197\frac{k}{L}$.

F_{ks} $\left[*\frac{L}{k}\right]$	F_{ku} $\left[*\frac{L}{k}\right]$	ΔF_k $\left[*\frac{L}{k}\right]$	$\tilde{\omega}_{ks}$ $\left[*\frac{ML^2}{k}\right]$	$\tilde{\omega}_{ku}$ $\left[*\frac{ML^2}{k}\right]$	$\Delta\tilde{\omega}_k$ $\left[*\frac{ML^2}{k}\right]$	$\tilde{\omega}_{kc}$ $\left[*\frac{ML^2}{k}\right]$	$F_{kc} = F_{ks} + \mu\Delta F_k$ $\left[*\frac{L}{k}\right]$
0.35	0.40	0.05	0.0147	-0.0083	-0.0230	2.0 e-5	0.3819
0.36	0.39	0.03	0.0101	-0.0037	-0.0138	6.1 e-6	0.3820
0.37	0.39	0.02	0.0055	-0.0037	-0.0092	1.3 e-6	0.3820
0.38	0.0383	0.003	0.0009	-0.0004	-0.0014	7.1 e-8	0.3820

When it is assumed that the buckling mode around the bifurcation point does not change, hence $\{\varphi(\tilde{\omega}_{ks})\} \approx \{\varphi(\tilde{\omega}_{ku})\} = \{\varphi\}$, one can estimate μ in $F_{kc} = F_{ks} + \mu\Delta F_k$ with:

$$\mu = -\frac{\{\varphi\}^T[K_t(\tilde{\omega}_{ks})]\{\varphi\}}{\{\varphi\}^T[\Delta K_t]\{\varphi\}} \qquad \textbf{eq. 19}$$

and $[\Delta K_t] = [K_t(\tilde{\omega}_{ku})] - [K_t(\tilde{\omega}_{ks})]$. This relation is the Rayleigh quotient which can be derived from the eigenvalue problem $([\Delta K_t] - \Delta\tilde{\omega}_k[M])\{\varphi\} = \{0\}$, with $\tilde{\omega}_{kc} = \tilde{\omega}_{ks} + \mu\Delta\tilde{\omega}_k = 0$,with $\mu \ge 0$ and with $\Delta\tilde{\omega}_k = \tilde{\omega}_{ku} - \tilde{\omega}_{ks} \le 0$.

The linearized undamped equations of motion of the Ziegler problem in matrix notation yields:

$$ML^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{Bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{Bmatrix} + k\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = FL\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix}$$

Changing the applied load will change the stiffness matrix and will result in a stiffness change $[\Delta K_t]$.

F_{ks} $\left[*\frac{L}{k}\right]$	F_{ku} $\left[*\frac{L}{k}\right]$	ΔF_k $\left[*\frac{L}{k}\right]$	$\{\varphi(\tilde{\omega}_{ks})\}$ $\frac{\varphi_2}{\varphi_1}$	$\{\varphi(\tilde{\omega}_{ku})\}$ $\frac{\varphi_2}{\varphi_1}$	$F_{kc} = F_{ks} + \mu\Delta F_k$ $\left[*\frac{L}{k}\right]$	$\tilde{\omega}_{kc}$	$\{\varphi(\tilde{\omega}_{kc})\}$ $\frac{\varphi_2}{\varphi_1}$
0.35	0.40	0.05	1.5972	1.6302	0.3820	-3.5 e-5	1.6182
0.36	0.39	0.03	1.6036	1.6234	0.3820	-1.7 e-5	1.6181
0.37	0.39	0.02	1.6101	1.6234	0.3820	-5.0 e-6	1.6180
0.38	0.0383	0.003	1.6167	1.6187	0.3820	-1.3 e-7	1.6180

The indirect search for a limit point

The indirect search for a limit point is not as straightforward as an indirect search for a bifurcation point. An equilibrium point just before and just after the limit point will give always lower applied loads than the load at the limit point. The limit point is often called the turning point [Seydel]. The equilibrium after the limit point is unstable. We require the limit point in order to obtain the buckling mode by means of the normal mode analysis with an eigenvalue $\tilde{\omega}_{kc} \rightarrow 0$. This limit point can be estimated with a curve fitting method and determining afterwards the maximum value of the applied load, being the critical load.
To this purpose we have to calculate equilibrium states before and after the limit point by using the continuation method (arc-length). The states after the limit point can be recognised because the number of negative terms on the main diagonal changes from zero to one. A second order and third order polynome curve fit can be made; the load F against a typical value of the displacement (rotation) q.

- Second order: $F = aq^2 + bq + c$. Three equilibrium points are needed, i.e. two points before and one point after the limit point
- Third order: $F = aq^3 + bq^2 + cq + d$. Three equilibrium points are needed, i.e. two points before and two points after the limit point

The coefficients a,b,c and d can be obtained by substituting the values for F and q. The limit point can be defined with: $\frac{dF}{dq} = 0$.

For x=0

F_{ks}	F_{ku}	ΔF_k	$\tilde{\omega}_{ks}$	$\tilde{\omega}_{ku}$	$\Delta\tilde{\omega}_k$	$F_{kc} = F_{ks} + \mu\Delta F_k$
0.96	1.02	0.06	0.1194	-0.0597	0.1791	0.9999

x	#	F 1/kL]	φ	3-order polynome F [1/kL]	4-order polynome F [1/kL]	$\tilde{\omega}_{kc}$ (3-order)	$\tilde{\omega}_{kc}$ (4-order)
0.1	1	0.6302	-0.1970	$F_{cr} = 0.6960$ $\varphi_{cr} = -0.3733$	$F_{cr} = 0.6951$ $\varphi_{cr} = -0.3733$	1.2667	0.1850
	2	0.6842	-0.2987				
	3	0.6945	-0.3997				
	4	0.6789	-0.5008				
0.2	1	0.5134	-0.3036	$F_{cr} = 0.5341$ $\varphi_{cr} = -0.4292$	$F_{cr} = 0.5339$ $\varphi_{cr} = -0.4245$	0.5672	0.2455
	2	0.5335	-0.4078				
	3	0.5251	-0.5120				
	4	0.4965	-0.6162				

The eigenvalues $\tilde{\omega}_{kc}$ at the limit points are obtained after the critical load had been reached, which is calculated with the load incrementation method instead of the arc-length method. In general the limit points are passed easily and the analysis proceeds in the unstable region until the defined load is obtained.

Real Problem, HPSA (Option 7) reboost loads

Design description HPSA, option 7

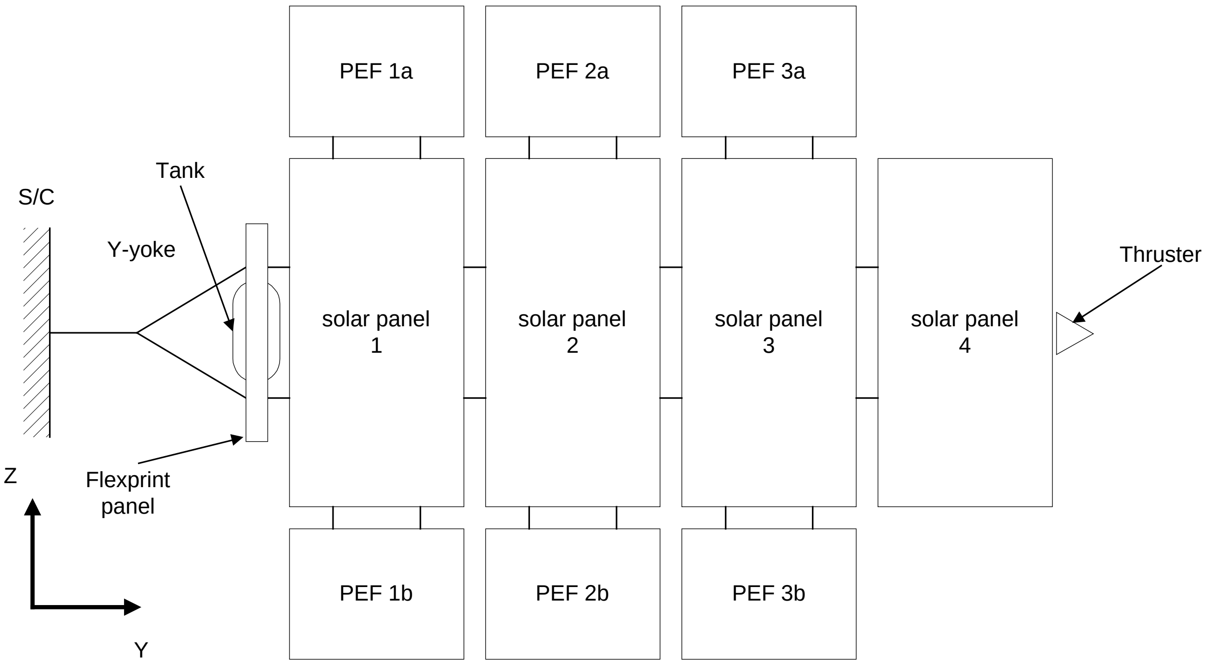
Goal:

Study to investigate feasibility of High Power Solar Array for TBD mission.

The relevant Mission and the HPSA solar Array requirements**:**

- Power requirements for two types of mission:
  1. 8 kW (EOL) and 11.5 kW (BOL)
  2. 15 kW (EOL) and 17 kW (BOL)
- Operating voltage: 100-120 V_{dc}
- Power to weight ratio: > 35 W/kg
- Frequency requirement:
  - Stowed: f_1 > 20 Hz (out-of-plane) and f_1 > 30 Hz (in-plane)
  - Deployed: f_1 > 0.05 Hz
- Reboost loads fully deployed (including dynamic amplification factor):
  - a = 0.021 g in S/C z-direction (Solar Array at angle γ)
  - α = 0.01 rad/s^2 about SADM axis

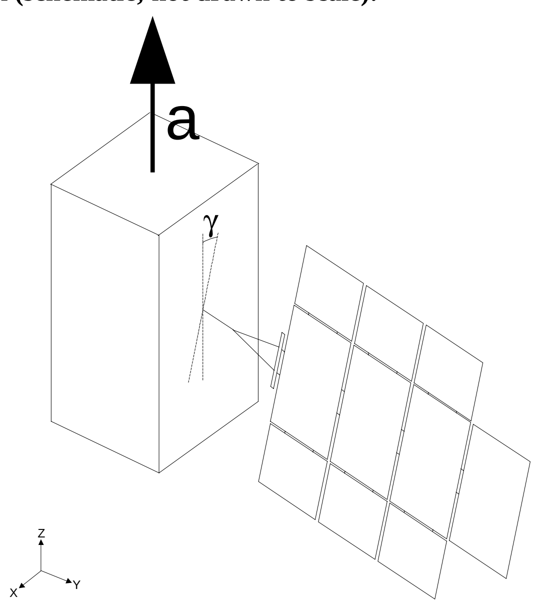
The schematic drawing of one wing of solar array:



The design description**:**

- Solar Array type: I-wing
- Yoke type: Y
- Number of panels per wing: 4
- Number of PEFs per wing: 6
- Panel dimensions: 4.3 m x 2.288 m
- PEF dimensions: 2.14 m x 2.288 m
- Total available area: 137.5 m^2
- Total mass of wing: 159 kg
- Type of solar cells: high η Si
- Power delivered: 16.5 kW (EOL)
- Additional payload: Thruster (16 kg) at wing tip; tank (100 kg) at yoke tip

Reboost acceleration situation (schematic; not drawn to scale):



The Finite Element Model of the HPSA

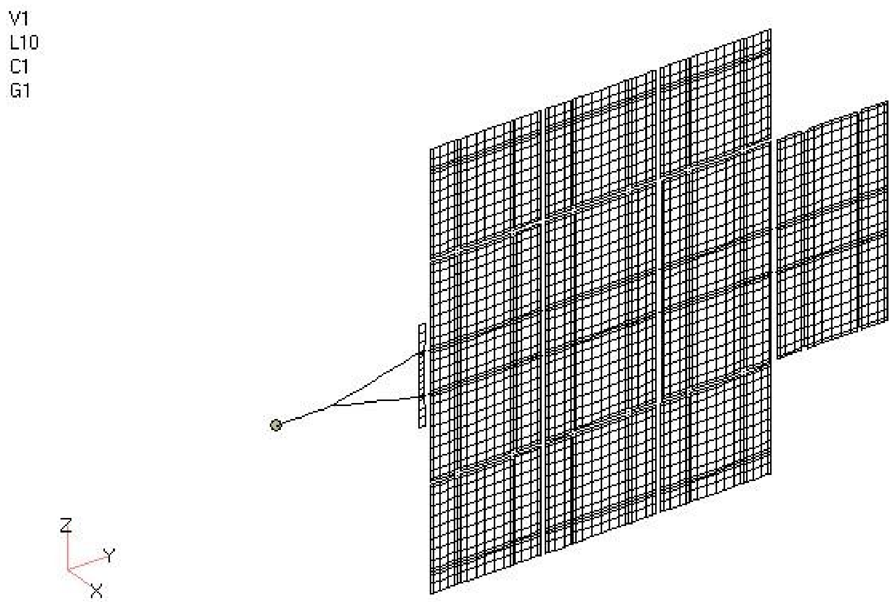


Figure 3 Complete FEM of HPSA

The NASTRAN finite element model (FEM) (Figure 3) of the deployed HPSA, is used to analyse its linear modal characteristics: natural frequencies, vibration modes and effective masses [Wijker]. The FEM contained many BAR and RBE elements, which are not supported in a GNL analysis. The BAR elements were replaced by BEAM elements and the RBEs were replaced by very stiff BAR elements and very stiff CELAS elements. Many CELAS elements are applied representing the stiffness of the locked hinges which interconnect the panels and the first panel to yoke.

After replacing the mentioned type of elements, a modal analysis has been carried out.

Mode #	Eigenvalue $\left[(\text{rad}/\text{s})^2\right]$	Natural Frequency [Hz]
1	0.1145	0.0539

After the modal analysis a linear buckling analysis had been performed. The results of the linear bifurcation analysis, using SOL 105, turned out to be meaningless. The lowest eigenvalue with respect to the reboost inertia load of 0.021 [g] in z-direction, was $\lambda = O(10^{-4})$.

During a GNL up to 0.5 [g] no bifurcation point had been encountered. However, the occurring stress may beyond the ultimate strength. Euler buckling was expected in the structural members of the yoke structure. However, these structural elements are mainly loaded in bending and in shear rather than in compression and tension.

After many hours of spend computer time it was decided to simplify the HPSA to a solar array with the original yoke and a smaller first panel.

To force the solar array into bifurcation an artificial simplified FEM had been created.

Simplified Model

The simplified FEM of the HPSA, the stripped version, is illustrated in Figure 4. The second area moment of inertia of the yoke was decreased in an attempt to introduce bifurcation into the more flexible yoke structure. The yoke and first panel are coupled with scalar elements which idealise the flexibility of the locked hinges.

The results of the linear bifurcation analysis using SOL 105 turned out to be meaningless. The lowest eigenvalue, with respect to the reboost inertia load of 0.021 [g] in z-direction, was in the $O(10^{-6})$.

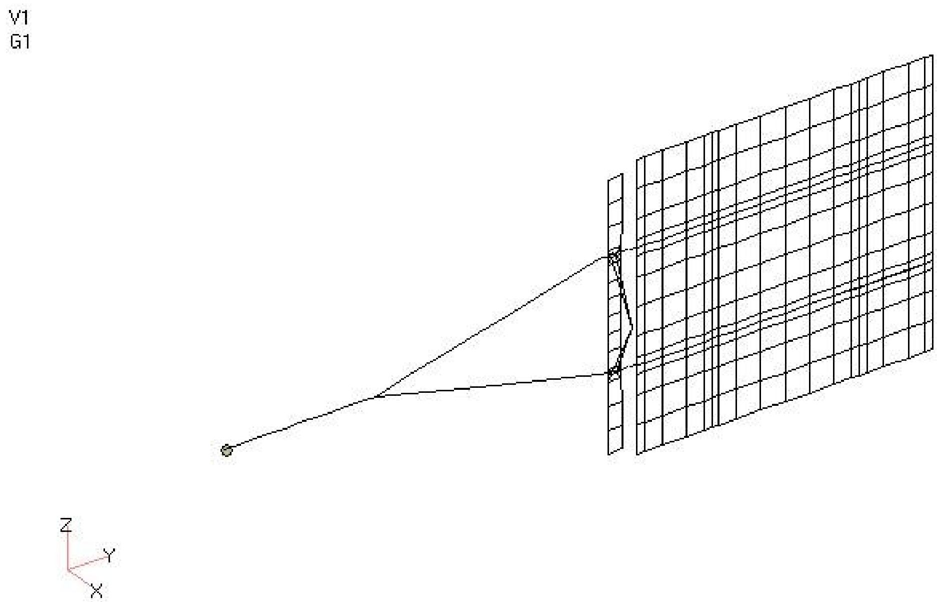
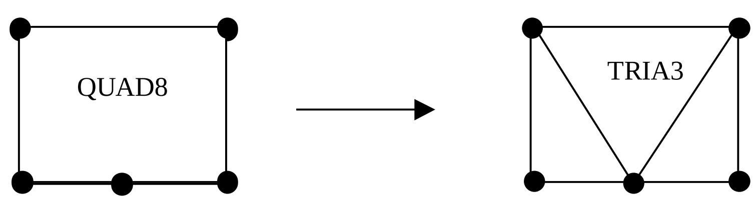


Figure 4 Simplified FEM of HPSA

The result of the modal analysis on the simplified model is given in the following table.

Mode #	Eigenvalue $[(rad/s)^2]$	Natural Frequency [Hz]
1	0.051198	0.036012

The modal analysis was followed by a GNL analysis, using SOL 106. After a few steps in the stable region the GNL analysis was switched into a modal analysis using the parameter NMLOOP. Very high negative eigenvalues were found. In two discrete areas in the neighbourhood of the hinges the associated “vibration modes” showed very high local deformations. This was caused by degenerated QUAD8 elements. Some QUAD8 elements consisted out of 5 nodes. So there was only one side with a mid node.



The QUAD8 elements as illustrated in above figure were replaced with three TRIA3 elements and the results of the linear bifurcation analysis became more reliable. The results are shown hereafter.

Linear Bifurcation (load –1g in minus Y-direction)

Mode #	Eigenvalue (Lowest Factor)
1	0.01498

Linear Bifurcation (load 0.021g in Z-direction)

Mode #	Eigenvalue (Lowest Factor)
1	$\pm$ 3.33525

The in plane reboost bifurcation load is very high. Although the bending stiffness of the structural members of the yoke were lowered one order, buckling was predicted at very high loads which were well beyond the strength capabilities of the structure.

In order to have an example with which the proposed procedure could be demonstrated, it was decided to analyse the simplified HPSA with in plane inertia loads in the negative y-directions (see Figure 4).

The approach to calculate the bifurcation or limit point was as follows:

- Start with the linear bifurcation analysis in order to determine the order of magnitude of the linear bifurcation load.
- The linear bifurcation load is the starting load for a GNL analysis either using the arc-length method or the load incrementation method. The latter shows very bad convergence characteristics in the neighbourhood of a bifurcation or limit point. In case of using the arc-length method the parameter TESTNEG must be activated. When a negative term appears on the factor main diagonal with PARAM, TESTNEG, -1 the solution procedure is aborted. The last load factor or load factors calculated during the last iteration step indicate the bifurcation or limit point.
- Investigate the equilibrium path in the neighbourhood of the estimated bifurcation or limit point. In this case the arc-length method is used; meaning that either the Riks' method or the Crisfield method must be used to pass bifurcation or limit points. These results must be saved into the NASTRAN restart database.
- Once a number of equilibrium states in the neighbourhood of the estimated unstable equilibrium are saved into the database, for every saved load (load factor, looped) the modal properties for the several load factors must be calculated using the tangent stiffness matrix and the (artificial) mass matrix. This mass matrix is not updated during the GNL analysis procedure. By passing the bifurcation or limit point, the eigenvalue of the dynamic eigenvalue problem (equation 8) will change sign. The bifurcation point or the limit point is estimated and the natural frequency and associated vibration mode (buckling mode) are calculated for that estimated point.

Load [g]	**Load factor** λ **(LOOPID)**	$\tilde{\omega}_{kc}$ $\left[(\mathrm{rad/s})^2\right]$	**Param Testneq**	**Remarks**
-0.01	1.000		-1	
-0.015	1.649		-1	After this step 2 negative terms in factor diagonal, unstable system, programme stops run
-0.01	1.000 (35)	0.01855	1	
-0.015	1.428 (58)	0.01402	1	After this step 1 negative term on factor diagonal
	1.456 (59)	0.01387	1	
	1.508 (62)	0.01355	1	
	1.642 (70)	0.01232	1	
	1.660 (71)	0.01217	1	
	2.000 (100)	0.01085	1	
-0.02	2.188 (105)	0.01097	1	
	2.267 (110)	0.01138	1	
	3.000 (165)	0.01776	1	

In the first run the param testneg was –1. The arc-length with the Crisfield method was used to solve the non linear equilibrium equations. The run aborts in subcase 2 after a load with load factor of 64.911% (inertia load is –0.01325 g). However, with 2 negative terms on the factor diagonal. The first two natural frequencies are close to each other.

In the second run the param testneq was set to 1. The run was divided into three subcases: in subcase 1 an inertia load up to –0.01 g, in the second subcase an inertia load up to –0.015g and in the third subcase 3 an inertia load up to –0.02 g was applied. The arc-length with the Crisfield method was used to solve the 165 non linear equilibrium states. After the load with a load factor of 42.771 in subcase 2 (inertia load – 0.01214 g) 1 negative term on the factor diagonal occurred. The deformation of the simplified solar array at a load with a load factor of 45.600 % (LOOPID=59) in subcase 2 (inertia load –0.01228 g) is illustrated in Figure 5.

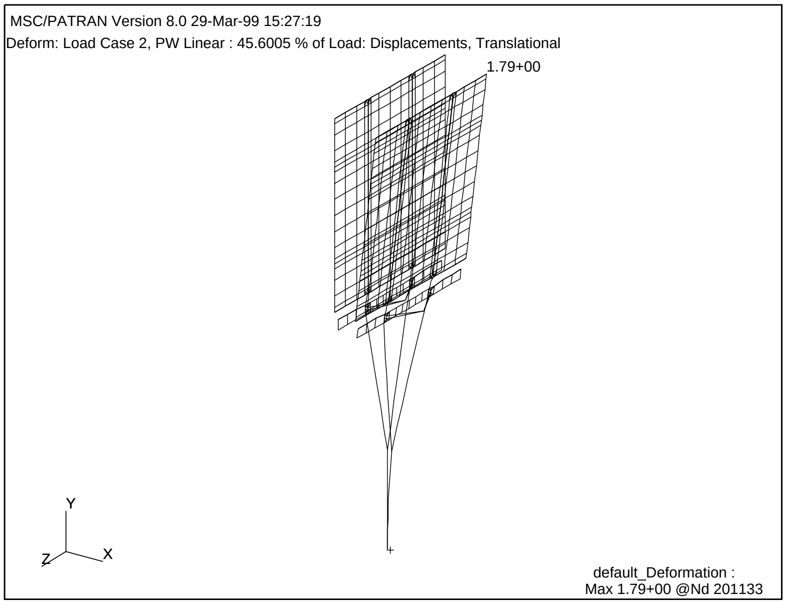


Figure 5 Deformation Load case 2 45.6% (LOOPID=59)

At the load with the load factor of 45.6% in subcase 2 the lowest eigenvalue of the dynamic system was calculated. A negative eigenvalue had been expected, however, this was not the case. The two lowest eigenvalues are $\tilde{\omega}_1 = 0.0138684$ and $\tilde{\omega}_2 = 0.0141186$. The mode shape with the lowest eigenvalue $\tilde{\omega}_1 = 0.0138684$ (f=0.01874 Hz) is illustrated in Figure 6.

It is not clear whether the mode shape at $\tilde{\omega}_1 = 0.0138684$ (f=0.01874 Hz) is also a buckling mode.
No negative eigenvalues are found, however, it was twice indicated that above the loads with load factors 1.649 and 1.428 negative terms are noticed.

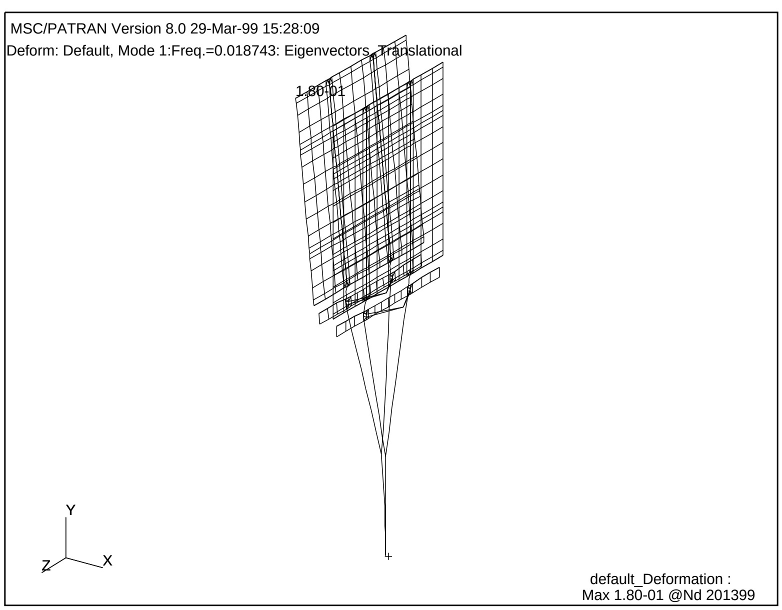


Figure 6 Mode at Load case 2 45.6% (NMLOOP=59)

Discussion

The dynamic approach to calculate the allowable buckling load and associated buckling mode with FEM's containing CELAS elements is demonstrated with simple examples. However, it is easier to locate a bifurcation point than limit points.

This means that by using the dynamic approach, the calculation of the buckling characteristics of geometrical perfect structures is very straight forward (reference Ziegler's problem). Euler buckling of bars is another example where bifurcation and associated buckling mode are calculated very conveniently.

Using the dynamic approach in case of imperfect structures (reference propped cantilever) will cause some problems. It is very difficult to determine the state of equilibrium where the natural frequency is zero. The neighbourhood of the bifurcation point and the limit point can be calculated. This value is used for the definition of the allowable load. The classical snap-through problem shows a very rapid change of the natural frequency in the vicinity of the limit point. The number of decimals after the decimal point has still a great influence on the natural frequencies. By using the arc-length method the limit point is passed easily. The load incrementation method will give better results in reaching the critical load and the calculation of the buckling mode afterwards.

The calculation of the bifurcation point and the limit point are not dependent on the mass distribution. However, the loads may be. With PARAM,WTMASS,m the lowest natural frequency can be increased artificially.

The arc-length method CRIS, RIKS in combination with SEMI is fast, because large load steps are possible.

Parameter NMLOOP is very convenient when the natural frequencies at a certain point of the path of equilibrium must be scanned.

The use of databases and restart capabilities in MSC/NASTRAN are strongly recommended:

- To restart in searching the bifurcation point or limit point
- To restart from the GNL analysis and continue with the modal analysis (PARAM,NMLOOP,n)

Conclusions

To avoid difficulties in buckling analyses with FEM's containing scalar elements, it is proposed to use a dynamic approach to calculate the buckling mode at either a bifurcation point or limit point. The following idea (procedure) is proposed to calculate the allowable buckling load with a FEM containing scalar spring elements:

- The bifurcation point or limit point of a loaded structural system is located with a standard Geometrical Non-Linearity (GNL) analysis. The continuation method (arc-length) is recommended for solving the non linear equations. Several available analysis methods must be tried; either the Riks method (RIKS) or the modified Riks method (MRIKS) or the Crisfield (CRIS) method. An unique recipe is not available.

- After a GNL analysis the corresponding buckling mode at the bifurcation point or limit point is calculated by means of a standard normal mode analysis using an adequate parameter setting (NMLOOP). Switching from a non linear analysis into a normal mode analysis does not seem to be possible.

The dynamic approach to calculate the allowable buckling load and associated buckling mode with FEM's containing CELAS elements is demonstrated with simple examples. However, a bifurcation point can be located more easily than limit points.

MSC/NASTRAN messages concerning negative terms on the factor diagonal shall be made more explicit in SOL 106. These messages are needed to locate the bifurcation or limit points.

Acknowledgements

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Abbreviations

BOL	Begin Of Life
S/C	Spacecraft, Satellite
EOL	End Of Life
FEM	Finite Element Model
GNL	Geometric Non-Linearity
HPSA	High Power Solar Array
PEF	Power Extension Flaps
SADM	Solar Array Drive mechanism
Si	Silicium
TBD	To Be Defined
TPE	Total Potential Energy
$[I]$	Identity matrix
KW	Kilo Watts
F	Applied constant load
$\{F_{cr,k}\}$	Critical load vector
F_{ks}	Stable Load
F_{ku}	Unstable Load
k	Spring stiffness
L	Length
$[M]$	Mass matrix
$(u), u_i$	Generalised displacement vector, *i*th component of displacement vector
(u_k^c)	*K*th Critical displacement vector
U	(Elastic) Strain Energy
W	Work of external conservative loads loads
$\omega_k, \tilde{\omega}_k, \lambda_k$	*k*th eigenvalue
$\tilde{\omega}_{ks}, \tilde{\omega}_{ku}, \tilde{\omega}_{ke}$	Eigenvales in stable , unstable equilibrium and bifurcation
η	Efficiency (high η Si cells)
φ	Angle
Π	Total Potential Energy of structure-load system $\Pi = U - W$
$\delta\Pi, \delta^2\Pi$	First, respectively second variation of the potential energy
$\|K\|$	Determinant of matrix K