STABILITY ANALYSIS OF PLATES AND DOUBLY CURVED SHALLOW SHELLS USING FINITE ELEMENT METHODS AND APPLICATIONS OF MSC/NASTRAN

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ABSTRACT :

*A variational formulation of doubly curved shallow shells is presented. The analysis used Reissner's two-field variable variational principle with the transverse displacement **w** and Airy stress function **F** as field variable. Euler-Lagrange equations and boundary conditions are obtained. A finite element based on this variational principle preserving $C^{(1)}$ continuity is formulated, and your eigenvalues for free vibrations and buckling analyses are abtained. Applications for free vibrations and buckling analysis in MSC/NASTRAN model are given as well as their respective geometry shape. Several numerical calculations are presented. The results obtained are discussed and are compared with previous analytical solutions and numerical calculations.*

1. INTRODUCTION

The finite element method is a numerical procedure for analyzing structures and continua. Usually the problem addressed is too complicated to be solved satisfactorily by classical analytical methods. The problem may concern stress analysis, heat conduction, or any of several other areas. The finite elements procedure produces many simultaneous algebraic equations, which are generated and solved on a digital computer. Finite element calculations are performed on personal computers, mainframes, and all sizes in between. Results are rarely exact. However, errors are decreased by processing more equations, and results accurate enough for engineering purposes are obtainable at reasonable cost.

The finite element method originated as a method of stress analysis. Today finite elements are also used to analyze problems of heat transfer, fluid flow, electric and magnetic fields, and many others. Structural analysis of plates and shells using two field variables, the transverse displacement w and the Airy stress function F, has been investigated by several authors. Reissner [1] introduced a variational principle with w and F being the field variables. Based on an order of magnitude analysis to justify the omission of the inplane inertial, Reissner [2] extended the application to dynamic analysis of shallow shells.

The structural dynamic analysis of plates and shells has usually been performed using the Hamilton's principle with the displacements u,v and w taken as the field variables of the problem. Alternatively, the problem can be formulated using a two field variable modified functional with the transverse displacement w and Airy stress function F, as the field variables of the problem. Bismarck-Nasr [3] presented a formulation using the transverse displacement w and Airy stress function F, as the field variables for the problem of free vibration analysis of isotropic cylindrically curved shallow shells. The Euler-Lagrange equations governing the problem and the boundary conditions were obtained, It was shown that the boundary conditions on F are simple and direct to apply as on w. The variational principle was used to derive a C^1 continuity rectangular finite element end numerical results of buckling analysis, free vibration for freely supported square curved panels and structural dynamic response problems of pre-loaded doubly curved isotropic shallow shells were presented.

Several numerical reuslts are presented and are compared with previous analytical solutions, numerical calculation, experimental findings and with MSC/Nastran models.

2. PROBLEM FORMULATION

The variational equation of thin cylindrically curved shallow shells and including the effect of the curvature of the shell in the transverse direction, the variational equation of an isotropic doubly curved shallow shells, considering the effect of the work done by an initial pre-load inplane prestress load, N_x , N_y and N_{xy} , can be write as,

$$d(K\text{-}U\text{-}U_i) = d(\frac{1}{2}\int_A rhw_{,t}^2 dA - \int_A w(\frac{F_{,xx}}{R_x} + \frac{F_{,yy}}{R_y})dA - \frac{D}{2}\int_A [w_{,xx}^2 + w_{,yy}^2 + 2nw_{,xx}w_{,yy} + 2(1-n)w_{,xy}^2]dA$$
$$+\frac{1}{2Eh}\int_A [F_{,xx}^2 + F_{,yy}^2 - 2nF_{,xx}F_{,yy} + 2(1+n)F_{,xy}^2]dA - \frac{1}{2}\int_A [N_{xx}w_{,x}^2 + N_{yy}w_{,y}^2 + 2N_{xy}w_{,x}w_{,y}]dA = 0 \quad \textbf{(1)}$$

where the transverse displacement *w* and Airy stress function *F*, are the functions subjected to variation. $D=Eh^3/12(1\text{-}n^2)$, *E* is Young's modulus, *h* is the shell thickness and ν is Poisson's ratio. The Airy stress function is defined as.

$$N_{xx} = \frac{r^2 F}{ry^2}, \qquad N_{yy} = \frac{r^2 F}{rx^2}, \qquad N_{xy} = -\frac{r^2 F}{rxry} \quad \textbf{(2)}$$

Performing the variational operation, grouping terms, and through application of Green's theorem, the Euler-Lagrange equations governing the problem are obtained and read,

$$D\nabla^4 w + \frac{F_{,xx}}{R_x} + \frac{F_{,yy}}{R_y} - N_{xx}w_{,xx} - N_{yy}w_{,yy} - 2N_{xy}w_{,xy} + rhw_{,tt} = 0$$
$$\frac{\nabla^4 F}{Eh} - \frac{w_{,xx}}{R_x} - \frac{w_{,yy}}{R_y} = 0$$

(3)

and the boundary condition for an edge ν = constant , are given by:

1 - Clamped edges $w = w_{,h} = 0$, and at a corner $F_{,hz} = 0$;

2 - Free edges $F = F_{,h} = 0$, and at a corner $M_{hz} = 0$ (or $w_{,hz} = 0$);

3 - Simply supported edges $w = 0$ and at a corner $F_{,hz} = 0$;

4 - Freely supported edges $w=F=0$.

A finite element solution for the problem at hand can be performed using rectangular elements preserving C^1 continuity based on the functional given in Eq.(1). Thus, we can write,

$$w(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2} H_{oi}(x)H_{oj}(y)w_{ij} + H_{1i}(x)H_{oj}(y)w_{,x_{ij}} + H_{oi}(x)H_{1j}(y)w_{,y_{ij}} + H_{1i}(x)H_{1j}(y)w_{,xy_{ij}}$$
$$F(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2} H_{oi}(x)H_{oj}(y)F_{ij} + H_{1i}(x)H_{oj}(y)F_{,x_{ij}} + H_{oi}(x)H_{1j}(y)F_{,y_{ij}} + H_{1i}(x)H_{1j}(y)F_{,xy_{ij}} \quad \textbf{(4)}$$

where H_{mn} are first order Hermitian polynomials. Using the standard finite elements technique we obtain for each element a set of two equations cast in the form below:

$$[K]_{ww}\{w\} + [K]_{wF}\{F\} + [N_{xx}[K_G]_{Nxx} + N_{yy}[K_G]_{Nyy} + N_{xy}[K_G]_{Nxy}]\{w\} + [M]_{ww}\{w_{,tt}\} = 0$$
$$[K]_{wF}^T\{w\} + [K]_{FF}\{F\} = 0$$

(5)

The element stiffness matrix [k_{ww}] is the same as the stiffness matrix of the sixteen degree of freedom plate bending element given by Bismarck-Nasr [8], the compatibility matrix [k_{FF}] is obtained from the stiffness matrix [k_{ww}] by changing the sign of ν and multiplying the whole matrix by 1/*DEh* The elements of the coupling matrix [k_{wF}] are given by,

$[K]_{wF}$ $$k_{ij_{wF}} = \frac{1}{R_x} S1_{\mathrm{a}}(n_j, n_{\mathrm{i}}) R2_{\mathrm{b}}(m_i, m_j) + \frac{1}{R_y} S1_{\mathrm{b}}(m_j, m_i) R2_{\mathrm{a}}(n_j, n_i) \quad \textbf{(6)}$$

where the (4x4) matrices $S1_a$ and $R2_b$ and element mass matrix are given by Bismarck-Nasr [8].

Using now the standard finite element assembly technique and applying the boundary conditions, we obtain for the whole structure the following two matrix equations,

$$[K]_{ww}\{w\} + [K]_{wF}\{F\} + [N_{xx}[K_G]_{Nxx} + N_{yy}[K_G]_{Nyy} + N_{xy}[K_G]_{Nxy}]\{w\} + [M]_{ww}\{w_{,tt}\} = 0$$
$$[K]_{wF}^T\{w\} + [K]_{FF}\{F\} = 0$$
(7)

We observe that the degree of freedom {F} can be eliminated using the compatibility equation of the system of equations, i.e., the second equation of the system (7), to obtain,

$$[[K]_{\mathrm{eq}} + N_{xx}[K_G]_{Nxx} + N_{yy}[K_G]_{Nyy} + N_{xy}[K_G]_{Nxy} - \omega^2[M]_{ww}]\{w\} = 0 \quad \textbf{(8)}$$

where,

$$[K]_{\mathrm{eq}} = [K]_{ww} - [K]_{wF}[K]_{FF}^{-1}[K]_{wF}^{\mathrm{T}} \quad \textbf{(9)}$$

An examination of Eq.(9) reveals that the computational effort required for the solution of the free vibration problem when the present formulation is used is equivalent to that of a flat plate. Further, the inplane boundary conditions are applied on F, $F_{,x}$,$F_{,y}$ and $F_{,xy}$ and are all nodal degrees of freedom of the finite element model.

3. NUMERICAL RESULTS AND DISCUSSIONS

The present formulation permits the studies of structural dynamic response problems of pre-loaded plates and shallow shells. As special cases are included the free vibration analysis of plates and shallow shells and the buckling analysis of plates and shallow shells. In the following some of the results obtained using the present formulation are reported and are compared with previous investigation whenever possible as well as MSC/Nastran model analysis.

3.1 - Buckling analysis of plates square simply supported and clamped on all edges

This tables show the results obtained using the present formulation and these are compared with previous analytical solutions and others finite elements formulations as well as MSC/Nastran model analysis.

Table 3.1.1. Buckling coefficients, $N_{cr}=N_{xx}a^2/p^2D$ simply supported on all edges.

Triangular Finite Element	Rectangular Finite Element

Timoshenko (exact)	Allman 1971	Clough 1968	Anderson 1968	Kapur e Hartz 1966	Dawe 1969	Carson 1969	This Work	MSC Nastran
Uniaxial (4,00)	4,031	4,126	3,72	3,770	3,978	4,001	4,222	4,421
Biaxial (2,00)	2,016	-	-	-	1,989	-	2,000	2,422
Shear (9,34)	10,131	-	-	-	9,481	9,418	9,416	9,523

Table 3.1.2. Buckling coefficients, $N_{cr}=N_{xx}a^2/p^2D$ clamped on all edges.

Triangular Finite Element				Rectangular Finite Element				
Timoshenko (exact)	Allman 1971	Clough 1968	Anderson 1968	Kapur e Hartz 1966	Dawe 1969	Carson 1969	This Work	MSC Nastran
Uniaxial (10,07)	10,990	-	9,30	9,284	10,147	-	10,192	11,043
Biaxial (5,30)	5,602	5,625	5,043	4,975	-	5,3271	5,326	6,357
Shear (14,71)	17,382	-	-	-	-	15,043	15,122	17,582

3.2 -Free vibration analysis of plates and cylindrically curved shallow shells

This tables show the results obtained using the present formulation to condition freely supported on all edges and these are compared with the Reissner analytical solution reported on 1955 and MSC/Nastran model analysis. From the results obtained, it can be observed that good accuracy was obtained using the present formulation with only a mesh size of 4 by 4 elements.

Table3.2.1. Natural frequency parameter of plate, $\Omega=\rho ha^4\omega^2/\pi^2D$
$a/b=1$, : $1/R_x = 1/R_y = 0$

mode	m	n	This work	MSC/Nastran	Reissner 1955
1	1	1	2,00932	2,00945	2,00000
2	1	2	5,00000	5,00023	5,00000
3	2	1	5,00000	5,00023	5,00000
4	2	2	8,00941	8,00992	8,00000
5	1	3	10,03123	10,06242	10,00000
6	3	1	10,03123	10,06242	10,00000
7	2	3	13,03223	13,06231	13,00000
8	3	2	13,03223	13,06231	13,00000
9	1	4	17,21627	17,32454	17,00000
10	4	1	17,21627	17,32454	17,00000

Table3.2.2.Natural frequency parameter of cylindrically curved shallow shells $\Omega=\rho ha^4\omega^2/\pi^2D$
$a/b=1$, : $1/R_x = 0{,}001\ cm^{-1}$; $1/R_y = 0$

mode	m	n	This work	MSC/Nastran	Reissner 1955
1	1	1	2,81532	2,95623	2,76444
2	1	2	5,10230	5,15632	5,05794
3	2	1	5,87379	5,98546	5,85865
4	2	2	8,22971	8,36954	8,22448
5	1	3	10,10363	10,11652	10,00728
6	3	1	10,40496	10,56231	10,57357
7	2	3	13,06285	13,07586	13,05294
8	3	2	13,50735	13,75423	13,26584
9	1	4	17,32312	17,36521	17,00148
10	4	1	17,70360	17,80523	17,37541

3.3-Presence of axial pre-load in the axial direction

The results presented are for free vibration analysis of cylindrically curved shallow shells for clamped conditions on all edges in the presence of axial pre-load in the axial direction.

Table 3.3.1- Fundamental natural frequency parameter, $\Omega=\rho h a^4\omega^2/\pi^2 D$ clamped for all edges
Cylindrically curved shallow shells with presence of an axial pre-load
$N^*=N_{xx}a^2/p^2D$, a/b=1 , $1/R$ = 0,01cm^{-1}

N^* kgf/cm	Ω
0	9,6
5	8,8
10	8,0
15	7,3
20	6,2
25	5,1
30	3,4
35	0,0

It is to be observed the critical loading condition is obtain as a subproduct of this analysis and is reached when a zero is observed.

4. CONCLUSIONS

The analysis presented permits the free vibration analysis, combined buckling loads and structural dynamic response in the presence of pre-loads.
It is show that the boundary conditions on Airy stress function are as simple and direct to apply as for the boundary conditions on the transverse displacement. The element used in the analysis is characterized by its high precision and direct application of the boundary conditions.
We showed to be Finite Elements Analysis a powerful tool in the study plates and shells stability.

ACKNOWLEDGMENTS

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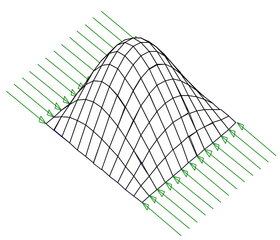
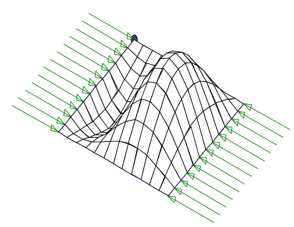
5. EXEMPLES SHAPES OBTAINED OF MSC/NASTRAN

Simply Supported, *m*=1, plate, uniaxial loading

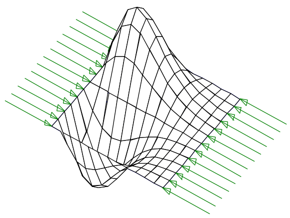
Buckling Coefficients



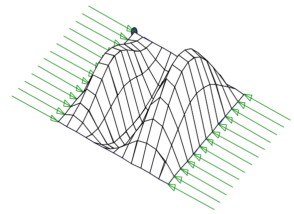
λ=4,421



λ=9,371



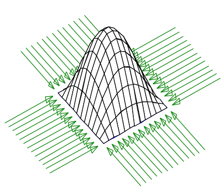
λ=14,656



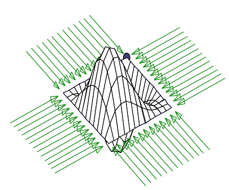
λ=17,343

Simply Supported, *m*=1, plate, biaxial loading

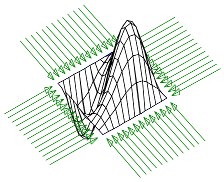
Buckling Coefficients



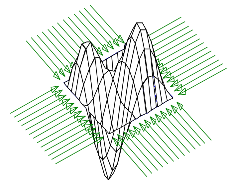
λ=2,422



λ=7,193



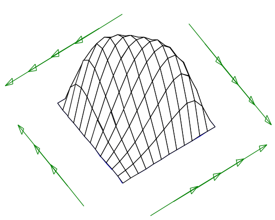
λ=8,024



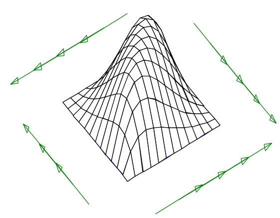
λ=12,301

Simply Supported, *m*=1, plate, shear loading

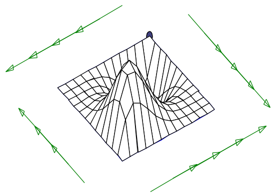
Buckling Coefficients



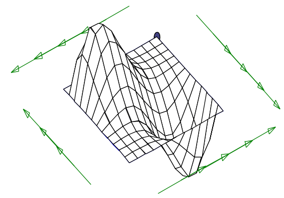
λ=9,523



λ=9,523



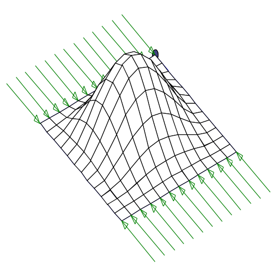
λ=13,152



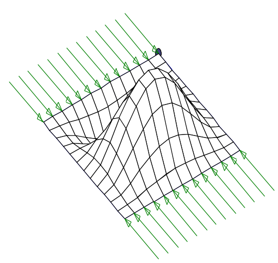
λ=13,152

Clamped, *m*=1, plate, uniaxial loading

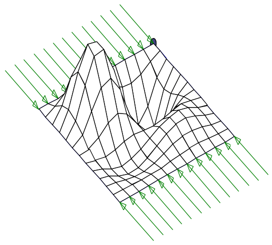
Buckling Coefficients



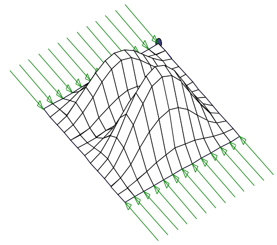
λ=11,043



λ=16,384



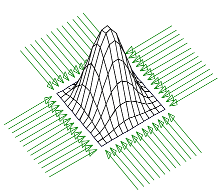
λ=27,235



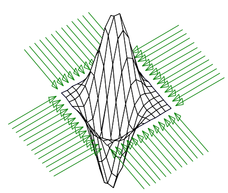
λ=30,497

Clamped, *m*=1, plate, biaxial loading

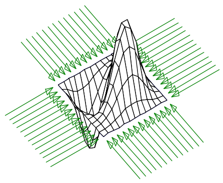
Buckling Coefficients



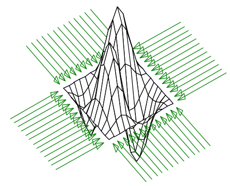
λ=6,357



λ=13,434



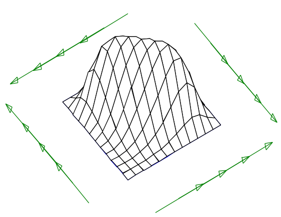
λ=14,689



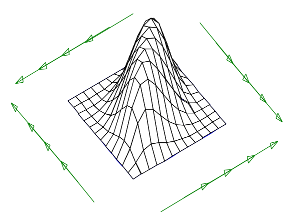
λ=20,263

Clamped, *m*=1, plate, shear loading

Buckling Coefficients



λ=17,582



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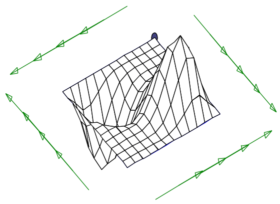
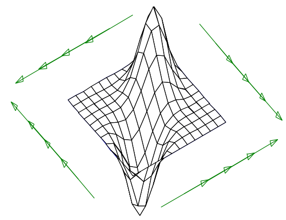
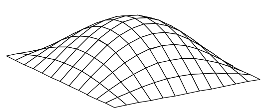


λ=22,308

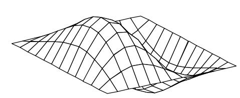


λ=22,308

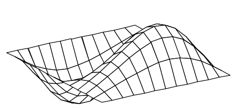
Freely Supported, *m*=1, plate - Natural frequency parameter



Ω=2,00954



Ω=5,00023



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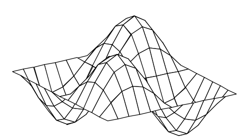
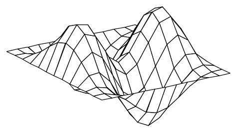
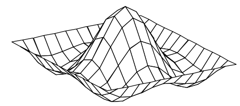
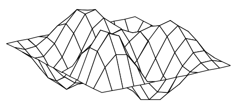
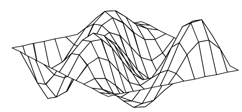
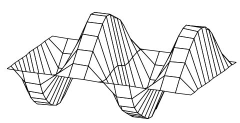
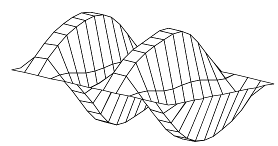
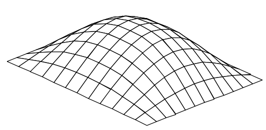
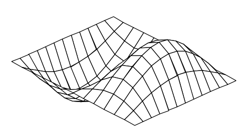
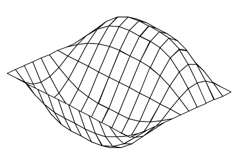
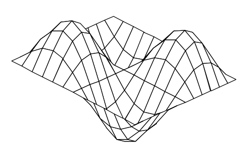
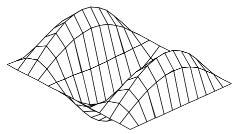


Ω=8,00992

 Ω=10,06242	 Ω=10,06242
 Ω=13,06231	 Ω=13,06231
 Ω=17,32454	 Ω=17,32454

Freely Supported, m=1, $\frac{1}{R_x} = 0$, $\frac{1}{R_y} = 0{,}001\,\mathrm{cm}^{-1}$ - Natural frequency parameter

 Ω=2,95623	 Ω=5,15632
 Ω=5,98546	 Ω=8,36954



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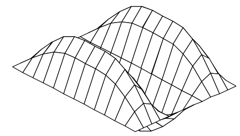

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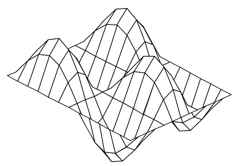

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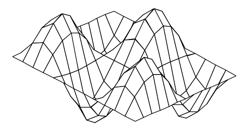

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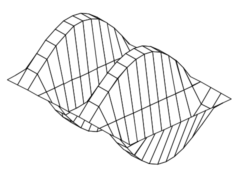
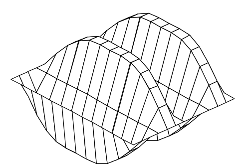


Ω=17,36521



Ω=17,80523

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