DETERMINATION OF NONLINEAR STIFFNESS COEFFICIENTS FOR FINITE ELEMENT MODELS WITH APPLICATION TO THE RANDOM VIBRATION PROBLEM

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Abstract

In this paper, a method for obtaining nonlinear stiffness coefficients in modal coordinates for geometrically nonlinear finite-element models is developed. The method requires application of a finite-element program with a geometrically nonlinear static capability. The MSC/NASTRAN code is employed for this purpose.

The equations of motion of a MDOF system are formulated in modal coordinates. A set of linear eigenvectors is used to approximate the solution of the nonlinear problem. The random vibration problem of the MDOF nonlinear system is then considered. The solutions obtained by application of two different versions of a stochastic linearization technique are compared with linear and exact (analytical) solutions in terms of root-mean-square (RMS) displacements and strains for a beam structure.

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Introduction

Development of the high speed flight vehicle technology necessitate the further theoretical development to understand the fatigue mechanisms and to estimate the service life of aerospace structures subjected to intense acoustic and thermal loads. Efforts to extend the performance and flight envelope of high speed aerospace vehicles have resulted in structures which may behave in a geometrically nonlinear fashion to the imposed loads.

Conventional (linear) prediction techniques can lead to grossly conservative designs and provide little understanding of the nonlinear behavior. A large body of work exists on the prediction of geometrically nonlinear dynamic response of structures. All methods currently in use are typically limited by their range of applicability or excessive computational expense.

Methods currently in use to predict geometrically nonlinear dynamic structural response include perturbation, Fokker-Plank-Kolmogorov (F-P-K), Monte Carlo simulation and stochastic linearization techniques. Perturbation techniques are limited to weak geometric nonlinearities. The F-P-K approach[1,2] yields exact solutions, but can only be applied to simple mechanical systems. Monte Carlo simulation is the most general method, but computational expense limits its applicability to rather simple structures. Finally, stochastic linearization methods (e.g. equivalent linearization, see[2-6]) have seen the most broad application for prediction of geometrically nonlinear dynamic response because of their ability to accurately capture the response statistics over a wide range of response levels while maintaining a relatively light computational burden.

The equations of motion of a MDOF, viscously damped geometrically nonlinear system can be written in the form

$$M\ddot{X} + C\dot{X} + KX + , (X) = F \qquad (1)$$

where M, C, K, X and F are the mass, damping, stiffness matrices, displacement response vector and the force excitation vector respectively. The vector function $, (X)$ generally includes 2nd and 3rd order terms in X. There exist mathematical difficulties in the derivation of a general solution to equation (1) for the case of random excitation. An approximate solution can be achieved by formation of an equivalent linear system:

$$M\ddot{X} + C\dot{X} + (K + K_e)X = F \qquad (2)$$

where K_e is the equivalent linear stiffness matrix. Assuming the Gaussian zero-mean loading and response, the convential stochastic linearization technique based on the force-error minimization (see for example Roberts et al.[3], Atalik et al.[4]) yields the following expression for the equivalent stiffness term (which replace the nonlinear one):

$$K_e = E[\frac{\partial ,}{\partial X}] \qquad (3)$$

where $E[...]$ is the operator of expectation of a random quantity.

section describes a method of determining the nonlinear stiffness coefficients through the use of the nonlinear static solution capability that exists in many commercial finite element codes.

For MDOF structures, it is expedient to seek a solution in modal coordinate space

$$X = \Phi q \tag{6}$$

where Φ is generally a subset $(L \leq N)$ of the linear eigenvectors (normal modes). Such a representation allows the size of the problem to be significantly reduced without a noticeable loss of accuracy in many cases.

From (1), one can obtain the following set of differential equations in terms of modal coordinates q_i $(i = 1, L)$:

$$\ddot{q}_i(t) + \sum_{j=1}^{L} c_{ij} \dot{q}_j(t) + k_i q_i(t) + \gamma_i(q_1, q_2, ..., q_L) = f_i(t) \tag{7}$$

where the nonlinear terms will be represented in the following form

$$\gamma_i(q_1, q_2, ..., q_L) = \sum_{j,k=j}^{L} a^i_{jk} q_j q_k + \sum_{j,k=j,l=k}^{L} b^i_{jkl} q_j q_k q_l \tag{8}$$

where the first index j takes values 1,2,...,L, the index k takes values from j (the current first index value) and up $(j + 1, j + 2 ...$ to $L)$ and the third index l takes values from k (the current second index value) and up $(k + 1,\ k + 2$ to $L)$.

A procedure for determination of the coefficients a^i_{jk} and b^i_{jkl} requires the application of a finite element program with a nonlinear static solution capability. In this study, the MSC/PATRAN and MSC/NASTRAN programs[8,9] are utilized.

The suggested technique is based on the restoration of nodal applied forces from enforced nodal displacements prescribed to the whole structure in a static solution (linear and nonlinear). Namely, by prescribing the physical nodal displacements (vector X_c) to the structure, one can restore the nodal forces F_T and the corresponding nonlinear contribution F_c:

$$F_c = , (X_c) = F_T - KX_c \tag{9}$$

The displacements X_c can be prescribed by creating a displacement constraint set for the model in MSC/PATRAN, then the nodal applied forces F_T will arise as single-point-constraint forces in a MSC/NASTRAN nonlinear static solution. In general, any set of displacements can be prescribed, e.g. for the bending problem of thin plate/beam-like structures, one can prescribe both out-of-plane and in-plane (membrane) components of displacement. The present analysis, however, is restricted to problems where in-plane displacements will be assumed small (thus neglected) and only the out-of-plane nonlinear stiffness coefficients are determined.

To illustrate the technique, one can begin with the prescription of displacements for the whole structure in the following form

$$X_c = \phi_1 q_1 \tag{10}$$

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The nodal force vectors F_T (nonlinear static solution) and $K X_c$ (linear static solution) are provided by MSC/NASTRAN. The nonlinear term F_c can then be evaluated by equation (9). The vector of modal forces $\tilde{F}_c = \Phi^T F_c$ is calculated and it is represented as

$$\tilde{F}_c = \Phi^T F_c = \Phi^T, (X_c) = \Phi^T, (\phi_1 q_1) = [a_{11}^i] q_1 q_1 + [b_{111}^i] q_1 q_1 q_1 \qquad (11)$$

where the sought stiffness coefficients $[a_{11}^i]$, $[b_{111}^i]$ are column-vectors $L \times 1$ $(i = 1, L)$. Note that all other nonlinear terms in (11) do not appear since $q_j = 0$ for $j \neq 1$.

Prescribing a displacement field with opposite sign $X_c = -\phi_1 q_1$ results in a modal force vector (denoted by $\tilde{F}_{-c}$):

$$\tilde{F}_{-c} = \Phi^T F_{-c} = \Phi^T, (X_c) = \Phi^T, (-\phi_1 q_1) = [a_{11}^i] q_1 q_1 - [b_{111}^i] q_1 q_1 q_1 \qquad (12)$$

where the quadratic (even) term will be the same as in (11) and the cubic (odd) term takes on a sign change.

Note that in the system of equations (11) and (12), the value of q_1 is given. The coefficients $[a_{11}^i]$, $[b_{111}^i]$ $(i = 1, L)$ can be determined from this system of $2 \times L$ linear equations. In an analogous manner, i.e. prescribing $X_c = \phi_j q_j$, all other coefficients $[a_{jj}^i]$, $[b_{jjj}^i]$ can be determined.

A similar technique can be employed to determine coefficients with two or three inequal lower indices, e.g., $[a_{12}^i]$, $[b_{112}^i]$, $[b_{122}^i]$ or $[b_{123}^i]$. Note that coefficients of the latter type appear only if the number of retained eigenvectors L in (6) is greater than or equal to 3. Determination of coefficients $[a_{12}^i]$, $[b_{112}^i]$ and $[b_{122}^i]$ will be considered as an example. Prescribe the displacement field to the model in the following form

$$X_c = \phi_1 q_1 + \phi_2 q_2$$

then the calculated (using MSC/NASTRAN) corresponding modal force vector $\tilde{F}_c$ is represented as follows

$$\tilde{F}_c = \Phi^T F_c = \Phi^T, (\phi_1 q_1 + \phi_2 q_2) = [a_{11}^i] q_1 q_1 + [b_{111}^i] q_1 q_1 q_1 + [a_{22}^i] q_2 q_2 + [b_{222}^i] q_2 q_2 q_2 +$$

$$[a_{12}^i] q_1 q_2 + [b_{112}^i] q_1 q_1 q_2 + [b_{122}^i] q_2 q_2 q_1 \qquad (13)$$

Prescribing the opposite sign displacement field

$$X_c = -\phi_1 q_1 - \phi_2 q_2$$

one obtains a second set of equations

$$\tilde{F}_{-c} = \Phi^T F_{-c} = \Phi^T, (-\phi_1 q_1 - \phi_2 q_2) = [a_{11}^i] q_1 q_1 - [b_{111}^i] q_1 q_1 q_1 + [a_{22}^i] q_2 q_2 -$$

$$[b_{222}^i] q_2 q_2 q_2 + [a_{12}^i] q_1 q_2 - [b_{112}^i] q_1 q_1 q_2 - [b_{122}^i] q_2 q_2 q_1 \qquad (14)$$

Summing (13) and (14), one obtains

$$\tilde{F}_c + \tilde{F}_{-c} = 2[a_{11}^i] q_1 q_1 + 2[a_{22}^i] q_2 q_2 + 2[a_{12}^i] q_1 q_2$$

From this equation, the coefficients $[a^i_{12}]$ are determined (note that the coefficients $[a^i_{11}]$, $[a^i_{22}]$ were already determined above).

Now we have two sets of L equations (13) and (14), but to determine cubic coefficients $[b^i_{112}]$ and $[b^i_{122}]$ from them is not possible since the system matrix has linearly dependent rows. Therefore, an additional type of displacement field is required. One can prescribe the following type

$$X_a = \phi_1 q_1 - \phi_2 q_2$$

Then the modal force vector is equal to

$$\tilde{F}_a = \Phi^T F_a = \Phi^T, (\phi_1 q_1 - \phi_2 q_2) = [a^i_{11}]q_1 q_1 + [b^i_{111}]q_1 q_1 q_1 + [a^i_{22}]q_2 q_2 - [b^i_{222}]q_2 q_2 q_2 -$$

$$[a^i_{12}]q_1 q_2 - [b^i_{112}]q_1 q_1 q_2 + [b^i_{122}]q_2 q_2 q_1 \tag{15}$$

From the system of $2 * L$ linear equations (13) and (15), the coefficients $[b^i_{112}]$ and $[b^i_{122}]$ can be determined. In a similar manner, all coefficients of the type $[b^i_{jjk}]$ and $[b^i_{kkj}]$ can be determined.

Now one can proceed with determination of coefficients with three different lower indices, like $[w^i_{123}]$. Prescribe a displacement field to the model in the following form

$$X_b = \phi_1 q_1 + \phi_2 q_2 + \phi_3 q_3$$

Calculated (by MSC/NASTRAN) the nonlinear nodal force vector F_b is

$$F_b = , (X_b) = , (\phi_1 q_1 + \phi_2 q_2 + \phi_3 q_3)$$

and the corresponding modal force vector will be

$$\tilde{F}_b = \Phi^T F_b = \Phi^T, (\phi_1 q_1 + \phi_2 q_2 + \phi_3 q_3) = [a^i_{11}]q_1 q_1 + [b^i_{111}]q_1 q_1 q_1 + [a^i_{22}]q_2 q_2$$

$$+[b^i_{222}]q_2 q_2 q_2 + [a^i_{33}]q_3 q_3 + [b^i_{333}]q_3 q_3 q_3 + [a^i_{12}]q_1 q_2 + [b^i_{112}]q_1 q_1 q_2 + [b^i_{221}]q_2 q_2 q_1 +$$

$$[a^i_{13}]q_1 q_3 + [b^i_{113}]q_1 q_1 q_3 + [b^i_{331}]q_3 q_3 q_1 + [a^i_{23}]q_2 q_3 + [b^i_{223}]q_2 q_2 q_3 + [b^i_{332}]q_3 q_3 q_2 + [b^i_{123}]q_1 q_2 q_3$$

Note that all coefficients in this equation have been already determined, except $[b^i_{123}]$. In analogous manner all coefficients of type $[b^i_{jkl}]$ ($j \neq k, j \neq l, k \neq l$) can be found.

Having the modal equations of motion (7) formulated, solution to these equations can now be undertaken through a variety of techniques. For the case of random loading, the application of the equivalent stochastic linearization was implemented in this study. Within the framework of the force-based technique, the equivalent stiffness matrix (according to the formula (3)) will have the following form

$$K_e = E[\frac{\partial(\gamma_1, \gamma_2, ..., \gamma_L)}{\partial(q_1, q_2, ..., q_L)}] \tag{16}$$

Note that the derivatives and expectations in (16) can be easily evaluated due to the analytical representation of the nonlinear terms in (8).

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Derivation of the potential energy function coefficients

In order to apply the energy-based version of stochastic linearization, it is necessary to have the expression of potential energy U of the system. Using expressions (8), one can proceed with the determination of the potential energy in terms of modal coordinates. It is known that nonlinear elastic force terms satisfy the following

$$\gamma_i(q_1, q_2, ..., q_L) = \frac{\partial U}{\partial q_i} \qquad i = 1, L \qquad (17)$$

where U is the potential energy generated by nonlinear terms only. Since all nonlinear coefficients in $\gamma_i(q_1, q_2, ..., q_L)$ have been determined, the potential energy function $U(q_1, q_2, ..., q_L)$ can be derived and it can be used in the energy-based stochastic linearization technique.

Represent the potential energy (nonlinear term contribution) of the system in the following form

$$U = \sum_{s,j,k,l}^{L} d_{sjkl} q_s q_j q_k q_l \qquad (18)$$

where the first index s takes values 1,2,...,L, the index j takes values from s (the current first index value) and up ($s+1, s+2$ ... to L), the third index k takes values from j (the current second index value) and up ($j + 1$, $j + 2$ to L), and the forth index l takes values from k (the current third index value) and up ($k + 1$, $k + 2$ to L). The task now is to find coefficients d_{sjkl}. Substituting (18) in (17) one obtains

$$\gamma_i(q_1, q_2, ..., q_L) = \sum_{j,k,l}^{L} b_{jkl}^{i} q_j q_k q_l = \frac{\partial}{\partial q_i} \Big(\sum_{s,j,k,l}^{L} d_{sjkl} q_s q_j q_k q_l \Big) \qquad (19)$$

Note that the quadratic terms in functions $\gamma_i(q_1, q_2, ..., q_L)$ were omitted, otherwise it would be necessary to introduce cubic terms in the expression (18). The quadratic terms in $\gamma_i(q_1, q_2, ..., q_L)$ arise in problems where membrane (in-plane) displacements of bending plate, beam-like structures occur. In this study the analysis is restricted to the flexural vibration problem with no in-plane components of the displacement.

From (19) the sought coefficients d_{sjkl} can be found as follows

$$d_{sjkl} = \begin{cases} b_{jkl}^{s}/2 & \text{if } s = j \\ b_{jkl}^{s}/3 & \text{if } s = j = k \\ b_{jkl}^{s}/4 & \text{if } s = j = k = l \\ b_{jkl}^{s} & \text{otherwise} \end{cases}$$

which can be easily verified by substitution of $U(q_1, ..., q_L)$ in the expressions (17).

Now when the expression for U in terms of modal coordinates has been derived one can proceed with application of energy-based version of linearization (see definition of the equivalent stiffness matrix in (4) for this case).

RMS Strain Calculation

When the solution of nonlinear equations of motion (7) is accomplished and the covariance matrix of displacements in physical coordinates is recovered, one can proceed with the determination of the variance/rms of strain components. Consider such a procedure of the strain determination for a flexural vibration problem of a beam in plane $X - Z$ (Fig. 1). The Green strain component ϵ_x at the point with the coordinates (x, z) is

$$\epsilon_x(x, z) = -z \frac{\partial^2 w(x)}{\partial x^2} + \frac{1}{2} (\frac{\partial w(x)}{\partial x})^2 \tag{20}$$

where w is the flexural displacement and the in-plane (membrane) component of displacement is not taken into account, since it was assumed that (for certain problems) it is small and can be neglected. Note that only the out-of-plane nonlinear stiffness coefficients were introduced in this study.

It is necessary to use the shape functions to represent the displacement and strain at any location of the beam. Considering two coordinates for each node w and θ (rotation about Y-axis), one can introduce four shape functions for each beam element, namely, $f_1(x)$, $f_2(x)$ associated with the 1st element node coordinates w_1, θ_1 and $f_3(x)$, $f_4(x)$ associated with the 2nd element node coordinates w_2, θ_2. These shape functions $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$ can be found, for example, in Weaver et al.[10]. Note that they are not necessary coincide with the NASTRAN's shape functions of the beam type element, but since the NASTRAN's shape functions are not available, these functions were employed.

Therefore assuming that for points within each element the flexural displacement is

$$w = w_1 f_1(x) + \theta_1 f_2(x) + w_2 f_3(x) + \theta_2 f_4(x)$$

from (20) one obtains

$$\epsilon_x(x, z) = -z(f_1^{''}(x)w_1 + f_2^{''}(x)\theta_1 + f_3^{''}(x)w_2 + f_4^{''}(x)\theta_2) + \frac{1}{2}(f_1^{'}(x)w_1 + f_2^{'}(x)\theta_1 +$$

$$f_3^{'}(x)w_2 + f_4^{'}(x)\theta_2)^2$$

and the expectation of ϵ_x^2 will be

$$E[\epsilon_x^2] = z^2(f_1^{''}f_1^{''}E[w_1^2] + 2f_1^{''}f_2^{''}E[w_1\theta_1] + 2f_1^{''}f_3^{''}E[w_1w_2] + 2f_1^{''}f_4^{''}E[w_1\theta_2] + f_2^{''}f_2^{''}E[\theta_1^2] +$$

$$2f_2^{''}f_3^{''}E[\theta_1 w_2] + 2f_2^{''}f_4^{''}E[\theta_1\theta_2] + f_3^{''}f_3^{''}E[w_2^2] + 2f_3^{''}f_4^{''}E[w_2\theta_2] + f_4^{''}f_4^{''}E[\theta_2^2]) +$$

$$\frac{1}{4}E[(f_1^{'}(x)w_1 + f_2^{'}(x)\theta_1 + f_3^{'}(x)w_2 + f_4^{'}(x)\theta_2)^4]$$

where the 3rd moments were omitted, since the Gaussian, zero-mean response is assumed. The terms with the co-factor $\frac{1}{4}$ produce moments of the 4th order which in turn, by (5), are expressed in terms of the 2nd order moments. Thus using the covariance matrix of displacements and shape functions of elements, one can calculate the variance of strain components (in this example $E[\epsilon_x^2]$) at any location of the structure and the root-mean-square (RMS) strain $\sqrt{E[\epsilon_x^2]}$ can be calculated as well.

Young's modulus	Poisson's ratio	density
0.73e+11	0.325	0.2763e+04
length	width	thickness
0.4572	0.0254	0.002261

Table 1: Parameters of the beam model

Numerical Results for a MDOF beam structure

The method which employs the technique described above, i.e. the determination of the nonlinear stiffness coefficients plus the two versions of stochastic linearization was implemented in a new in-house code.

The numerical results presented in this section correspond to a clamped-clamped beam model in Figure 1. The finite-element model had 19 nodes and 18 beam elements. The parameters of the model in Fig. 1 are shown in Table 1 (system of units is SI, $[m]$, $[N/m^2]$, $[kg]$ etc.), where the width and thickness are dimensions of the cross-section of the beam. The first two natural frequencies (associated with flexural symmetric modes in the excitation plane) are 57.4 Hz and 310.1 Hz. These two mass-normalized modes were chosen to approximate the motion according to formula (6).

The nonlinear stiffness coefficients determined with application of the procedure described above are summarized in Table 2. The quadratic terms were negligible, so only the 3rd order terms are shown. Since the modal coordinates q_1, q_2 are nondimensional, the units of these nonlinear coefficients are in $[N * m]$.

Note that from (17), it would follow that

$$\frac{\partial \gamma_j}{\partial q_k} = \frac{\partial \gamma_k}{\partial q_j} = \frac{\partial^2 U}{\partial q_k q_j}$$

Comparing the terms with like powers in q_j and q_k leads to a certain relation between the nonlinear coefficients, for example, for the cubic coefficients b^1_{122} and b^2_{112} it is

$$b^1_{122} = b^2_{112}$$

and for other types, it is

$$3b^1_{222} = b^2_{122} \qquad 3b^2_{111} = b^1_{112}$$

It turned out that the computed nonlinear stiffness coefficients (see Table 2) are in an excellent agreement with these relations.

The RMS flexural displacements and RMS strain component ϵ_x of the points located on the surface layer of the beam and along the coordinate x were obtained by using the force-based and energy-based versions of stochastic linearization and results are compared with the linear solution and with exact (FPK) solution for some cases below.

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b_{111}^1	b_{222}^1	b_{112}^1	b_{122}^1
0.899e+12	0.977e+13	0.191e+13	0.139e+14
b_{111}^2	b_{222}^2	b_{112}^2	b_{122}^2
0.638e+12	0.608e+14	0.139e+14	0.293e+14

Table 2: Nonlinear stiffness coefficients for the beam model

The system stiffness and damping matrices of the beam model in modal mass-normalized coordinates were as follows

$$K = \begin{bmatrix} 1.30098E + 05 & 0 \\ 0 & 3.79653E + 06 \end{bmatrix} \qquad C = \begin{bmatrix} 4.039 & 0 \\ 0 & 4.039 \end{bmatrix}$$

At first, the following type of the spectral density matrix of the external forces (in modal coordinates) was assumed:

$$S_f(\omega) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

which constituted a white noise excitation with the frequency band taken from 0 to 550 Hz. For this type of excitation the FPK solution was possible to obtain, see, for example Bolotin[1]. The RMS flexural displacements as function of coordinate x are shown in Fig. 2, where due to the symmetry only a half of the beam is shown. One can see that the force-based version slightly underpredicts the values of RMS displacements and that the linear analysis overpredicts it by about 50 % (at the middle node 10). RMS displacements at the middle node for the nonlinear analyses achieve about 60 % of the beam's thickness.

The RMS strains shown in Figures 3 and 4 indicate that FPK solution is between the force-based and energy-based solutions for all points along the beam and for both cases of loading. For the linear analysis (with the term , $(X)=0$ in (1)) the linear strain recovery was applied, i.e. the 2nd term in (20) was omitted.

As a 2nd type of excitation, a uniform (in space) white noise type pressure was applied to the beam. The frequency band was the same 0–550 Hz. This type of excitation corresponded to the following spectral density matrix of forces in modal coordinates:

$$S_f(\omega) = \begin{bmatrix} 0.0536 & -0.0236 \\ -0.0236 & 0.01052 \end{bmatrix}$$

where the units are in $[\frac{(N*m)*(N*m)}{rad/s}]$. The FPK solution was not available for this type of loading. The RMS strains for this case are shown in Fig. 5. One can see a larger difference between the two versions of linearization. The linear analysis predicts larger values of maximum strains than the nonlinear analyses and underpredicts strains for some portion of the beam. The difference in maximum RMS strain (at the clamping) between the linear and nonlinear analyses achieves about 50 %.

One can see that in all cases the energy-based version of linearization yields greater values of strain (achieving 15 % difference in some cases) than the values

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from the force-based version, i.e. it produces more conservative estimate for RMS strains.

Conclusions

A new method for determination of nonlinear stiffness coefficients has been suggested which utilizes a finite-element commercial software with geometrically nonlinear static capability. It has been shown that application of MSC/NASTRAN and MSC/PATRAN for this purpose is sufficient.

This method has been incorporated into a program which calculates a steady-state response of a MDOF structure to a Gaussian zero-mean random excitation. Efforts are presently underway to implement this method into MSC/NASTRAN through a DMAP Alter.

Two versions of the stochastic linearization have been compared on example of beam structure and the results have shown that the energy-based version (in all cases) provided a more conservative estimate of strains than the conventional (force-based) version. This is a useful fact, since application of the energy-based version would lead to more safe estimate of the fatigue life of the structure.

Acknowledgement

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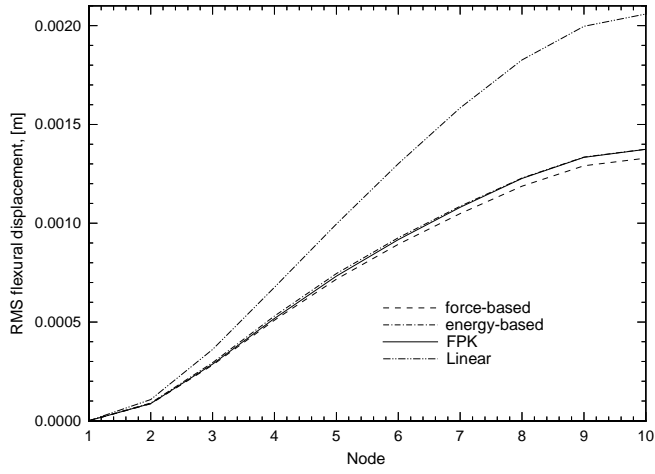
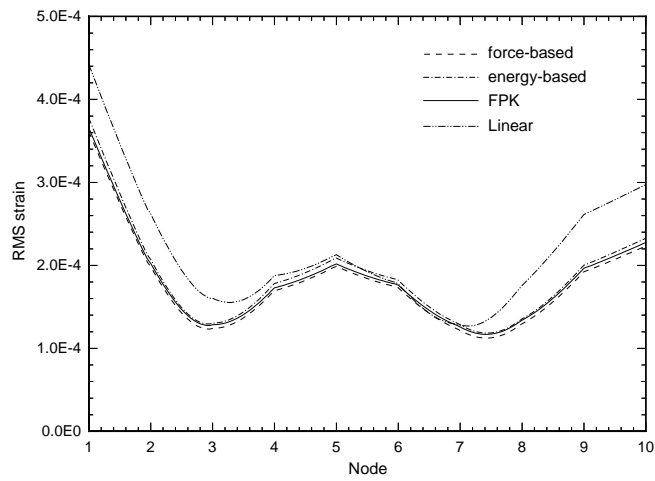Figure 2: RMS displacement w vs. longitudinal coordinate of the beam for a=0.02 $\frac{(N*m)*(N*m)}{rad/s}$

Figure 3: RMS strain ϵ_x vs. longitudinal coordinate of the beam for a=0.02 $\frac{(N*m)*(N*m)}{rad/s}$
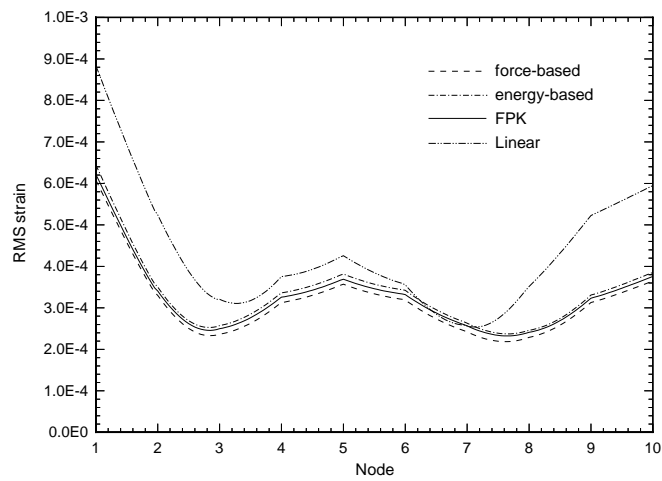
Figure 4: RMS strain ϵ_x vs. longitudinal coordinate of the beam for a=0.08 $\frac{(N*m)*(N*m)}{rad/s}$
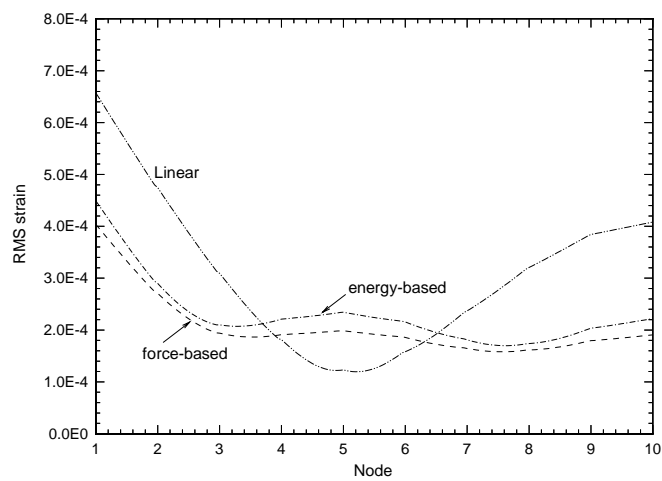

Figure 5: RMS strain ϵ_x vs. longitudinal coordinate of the beam. Uniform (in space) white-noise pressure