EXTENSIONS TO MSC/NASTRAN TO SOLVE FLEXIBLE ROTOR PROBLEMS*

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ABSTRACT

The general dynamic equations associated with rotating flexible bodies differ from those used to analyze nonrotating structures in a number of ways. Besides accommodating the usual mass and stiffness matrices, an appropriate finite element code for such problems must be capable of including Coriolis and centrifugal acceleration terms. These latter terms arise from the use of a rotating coordinate system which nominally follows the flexible body as it rotates. If part of the structure does not rotate, such as with a rotor housing, the computer program must also accommodate multiple rotating and nonrotating coordinate systems. In this paper, the additional rotor acceleration terms are derived and their implications are discussed from both a physical and computational point of view. Particular attention is devoted to flexible bearing, dynamic shift of initial imbalance, Coriolis and centrifugal effects. Interpretations of the steady-state forces on both the rotor and housing are presented and the requirements for using the MacNeal-Schwendler Corporation modified version of NASTRAN (MSC/NASTRAN) on such problems are given. In addition, a low speed iterative solution approach is described and tested on simple examples for which the exact solutions are known. By comparing the known exact solutions with the approximate ones, the proposed iterative solution accuracy and convergence rates are heuristically inferred.

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INTRODUCTION AND SUMMARY

The structural dynamic analysis of a 300 lb shaft rotating at 1500 rpm was required to help in the design of a precision test device. The test unit, which is called a Spin Mount System (SMS) and is shown schematically in Figure 1, will be used to spin up and help measure extremely small surface deviations in certain smaller optical systems. Since the SMS is required to help detect off-axis motions of the order of several seconds of arc (an arc-second is 4.85 microradians), it must be balanced nearly perfectly, have extremely smooth running bearings, and produce a minimum of motor and friction torque ripple.

The following effort was undertaken to support the design and development phase of the SMS because standard finite element formulations and computer codes do not consider rotational effects explicitly. This paper presents the derivation of the governing linearized equations, interprets the additional terms caused by spin, and discusses modifications to the standard MacNeal-Schwendler Corporation version of NASTRAN (MSC/NASTRAN) formulation to include rotation terms. Special attention is also devoted in this work to considerations of the dynamic disturbances, stability and low speed solution approximations.

The reader may also wish to consult several other derivations of the flexible rotor equations such as those contained in References 1-4. All of these touch upon various aspects of the rotor problem, some of which were not important for the present study. However, the potential user of such works should proceed in an extremely cautious manner since most of these papers leave out various terms from those included herein, and justifications for deleting such terms should be questioned in light of the intended application.

DERIVATION OF EQUATIONS

For the purpose of deriving the governing dynamic equations for a flexible rotor and housing assembly, such as the SMS of Figure 1, it is convenient to select a rotating coordinate system for the rotor and a stationary one for the housing. The reason for this is that the location of

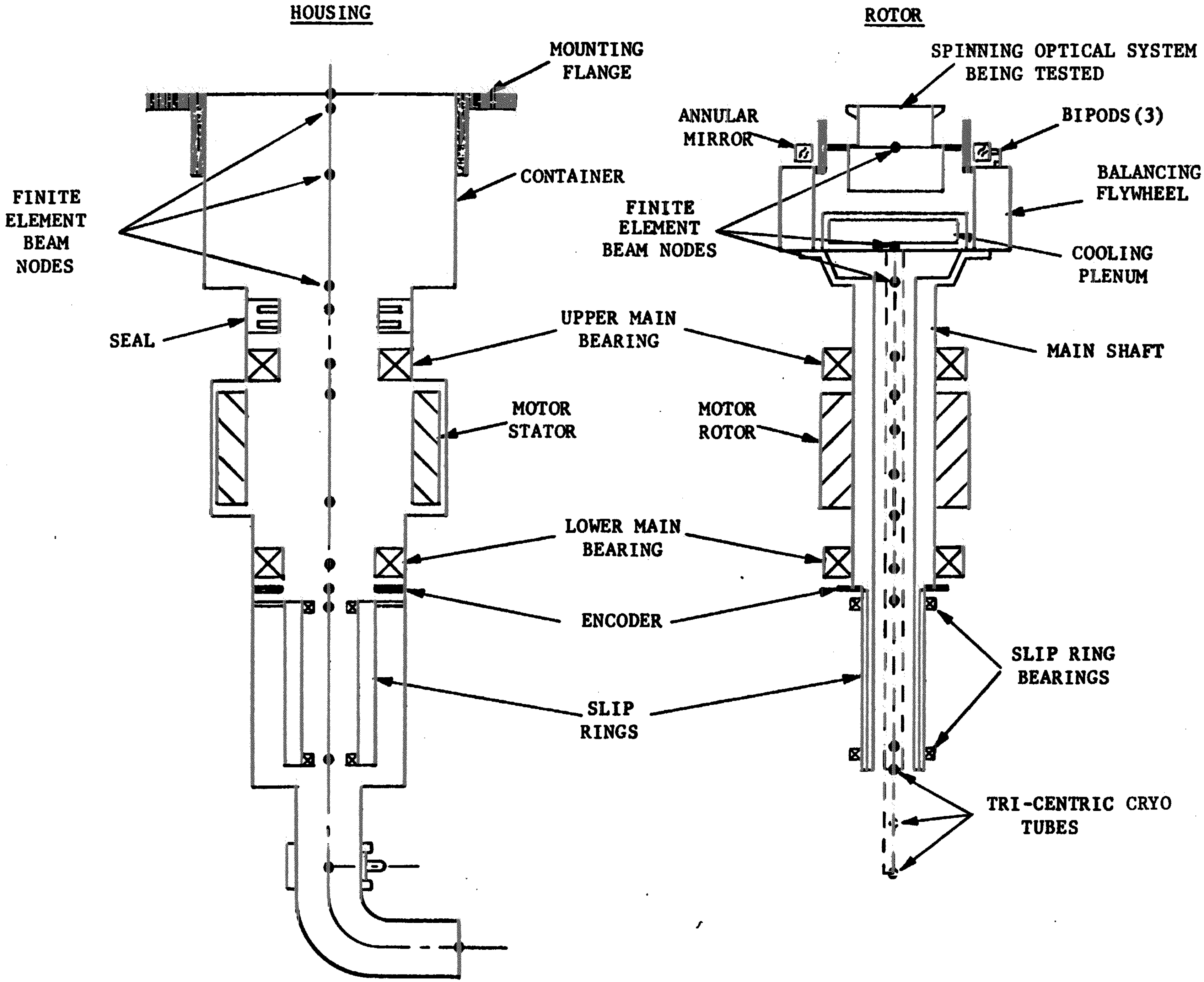


Figure 1. Spin Mount System Dynamic Model

an element of mass of fixed identity would otherwise continuously change its coordinates and be difficult to track. In addition, the associated dynamic equations would be difficult to linearize.

A simplifying concept that is adopted for purposes of the present derivation is to assume that all inertia properties are concentrated, or lumped, at a finite number of discrete points (called nodes). Thus, the translational and rotational equations are derived for a typical node (i) and then assembled to treat the entire system.

Housing

With regard to the nonrotating structural housing, the lumped parameter linear equations of dynamic motion, combined with a finite element formulation, result in the usual matrix equation:

$$[M_H]\{\ddot{\delta}_H\} + [D_H]\{\dot{\delta}_H\} + [K_H]\{\delta_H\} = \{F_h\} + \{F_{HB}\} \quad (1)$$

where $[M_H]$ encompasses all the nodal masses and mass moments of inertia for the housing, $[D_H]$ is the housing's damping matrix, $[K_H]$ is the housing stiffness matrix, and $\{\delta H\}$ is the displacement vector.

The force vectors represent external forces on the housing, $\{F_H\}$, and housing bearing loads, $\{F_{HB}\}$, induced by the rotation of the rotor. Since this matrix equation is well known and requires no further discussion, we will turn our main attention to a derivation of the flexible rotor equations. These require special consideration and yield additional terms in the dynamic formulation.

Rotor Translation

The basic vector equation for translational motion of a single mass, m_i, in a uniformly rotating coordinate system is

$$m_i\left[\ddot{\vec{\delta}}'_i + 2\,\vec{\Omega}_o \times \dot{\vec{\delta}}'_i + \vec{\Omega}_o \times \{\vec{\Omega}_o \times (\vec{\rho}_i + \vec{\delta}'_i)\}\right] = \vec{F}_i \quad (2)$$

in which m_i is the mass associated with the i^{th} rotor node, δ'_i is the deflection relative to the coordinate system which rotates with constant speed, ω_o, about the original shaft axis, $\vec{\rho}_i$ is the initial static offset of the mass center from the spin (x) axis, and $\vec{F}_i$ is the net vector force on node i.

The fact that $\vec{\Omega}_o$, the coordinate system's angular velocity, is a constant has been emphasized (by underlining) in the previous paragraph. This is not to say that the mass necessarily rotates at a constant angular velocity. However, it is assumed that $\vec{\rho}_i$, the offset position of the mass center, is extremely small compared to the radius of gyration of m_i about the x axis, and its variation with respect to the instantaneous rotor angular velocity, ($\vec{\Omega}$), may be neglected in Equation (2). The reason for these comments will become clearer when we consider the i^{th} node's rotational dynamic equations.

The vector cross product, $\vec{A} \times \vec{B}$, may be represented in matrix form by letting

$$\vec{A} \times \vec{B} = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \vec{B} \tag{3}$$

where

$$\vec{A} \equiv \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} \tag{4}$$

Performing the cross product operations in Equation (2), by the manner suggested in Equation (3), results in the vector equation

$$m_i \ddot{\vec{\delta}}'_i + 2m_i\omega_o \, [\beta]\dot{\vec{\delta}}'_i + m_i\omega_o^2 \, [\beta^2] \, \vec{\delta}'_i = \vec{F}_i - m_i\omega_o^2 \, [\beta^2] \, \vec{\rho}_i \tag{5}$$

where

$$[\beta] \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (6)$$

and

$$[\beta^2] \equiv [\beta]\,[\beta] = - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotor Angular Motion

The governing dynamic vector equation for rotational motion at node i, in rotating coordinates, is

$$\vec{T}_i = \frac{d\vec{H}_i}{dt} + \vec{\Omega}_i \times \vec{H}_i \quad (7)$$

where

$\vec{T}_i$ is the vector nodal torque

$\vec{H}_i$ is the nodal angular momentum vector given in terms of the constant centroidal inertia matrix $[I_i]$ as:

$$\vec{H}_i = [I_i]\,\vec{\Omega}_i \quad (8)$$

and $\vec{\Omega}_i$, is the instantaneous nodal rotational velocity, of m_i as well as the angular velocity of the rotating coordinate systems (note that it is not necessarily equal to $\vec{\Omega}_o$).

If we had not used $\vec{\Omega}_i$ as the local coordinate system angular velocity, $[I_i]$ would constantly change with respect to the coordinates used to calculate it. As noted earlier, this complication was not necessary earlier (for Equation (2)) since the mass center off-set, $\vec{\rho}_i$, was assumed small relative to the radius of gyration of the rotor. If this were not the case (as is true for structures which are not basically symmetric about the axis of rotation, e.g., helicopter blades), Equation (2) would require additional terms to accurately describe the translational motions, $\vec{\delta}_i'$.

Linearization of $\vec{\Omega}$ about the nominal spin velocity $\vec{\Omega}_o$ gives (see Figures 2.a and 2.b):

$$\vec{\Omega} = \vec{\Omega}_o + \begin{Bmatrix} \dot{\theta}_x \\ \dot{\theta}_y - \omega_o \theta_z \\ \dot{\theta}_z + \omega_o \theta_y \end{Bmatrix} \tag{9}$$

where

$$\vec{\Omega}_o = \begin{Bmatrix} \omega_o \\ 0 \\ 0 \end{Bmatrix} \tag{10}$$

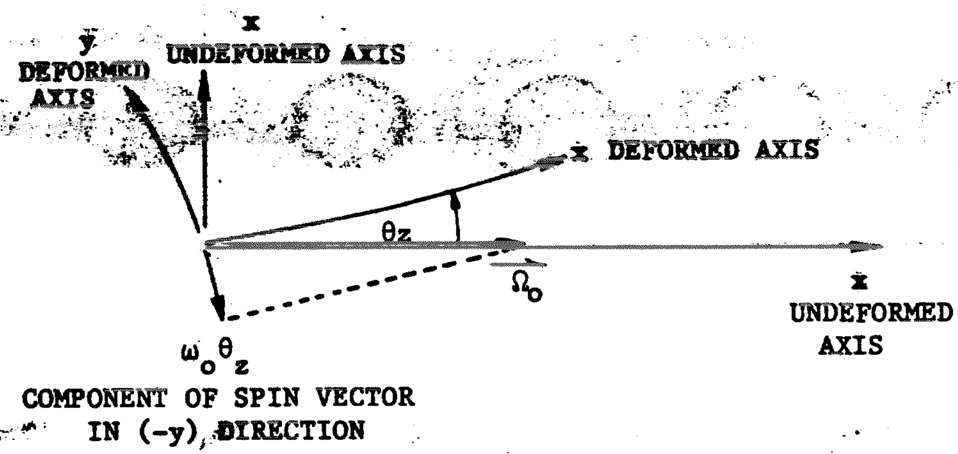
Equation (9) may be rewritten as

$$\vec{\Omega} = \vec{\Omega}_o + \dot{\vec{\theta}} + \omega_o [\beta] \vec{\theta} \tag{11}$$

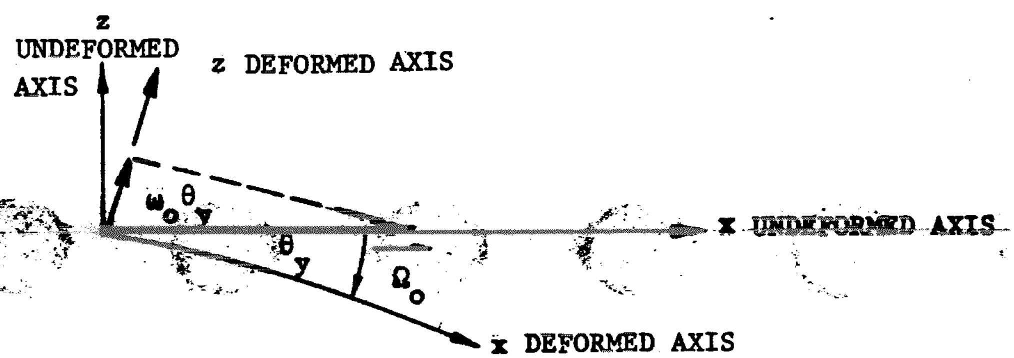
where

$$\vec{\theta} \equiv \begin{Bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} \tag{12}$$

Substituting Equation (11) into Equations (7) and (8), and linearizing the result, yields (after some manipulation),



a. x-y PLANE



b. x-z PLANE

Figure 2. Linearized Components of Spin Vector in Deformed Rotor Coordinate System

$$\vec{T}_i = [I_i]\,\ddot{\vec{\theta}}_i + \omega_o \left|[I_i][\beta] + [\beta][I_i] - \omega_o' \left([I_i]\,\vec{\Omega}_o\right) x\right| \dot{\vec{\theta}}_i$$

$$+ \omega_o^2 \left|[\beta]\ [I_i][\beta] - \omega_o \left([I_i]\,\vec{\Omega}_o\right) x\right| \vec{\theta}_i \tag{13}$$

$$- \omega_o\ [\beta]\ [I_i]\,\vec{\Omega}_o$$

Substituting the elements of the inertia matrix into Equation (13) and performing the matrix operations involving $[I_i]$, $[\beta]$ and $\vec{\Omega}_o$, combining terms and premultiplying by the matrix $[r]_i$ (where:

$$[r]_i \equiv \begin{bmatrix} 1 & -\theta_z & \theta_y \\ \theta_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_i \tag{14}$$

symmetricized the $[C_i']$ matrix), and linearizing yields:

$$[I_i]\,\ddot{\vec{\theta}}_i + 2\omega_o[G_i']\,\dot{\vec{\theta}}_i - \omega_o^2\ [C_i']\,\vec{\theta}_i = \vec{T}_i' + \omega_o^2\,\vec{P}_i \tag{15}$$

where

$$\vec{P}_i = \begin{Bmatrix} 0 \\ -P_{xz} \\ P_{xy} \end{Bmatrix} \tag{16}$$

$$[I_i] = \begin{bmatrix} I_x & -P_{xy} & -P_{xz} \\ -P_{xy} & I_y & -P_{yz} \\ -P_{xz} & -P_{yz} & I_z \end{bmatrix}_i \tag{17}$$

$$[G_i'] = \begin{bmatrix} 0 & -P_{zx} & P_{xy} \\ P_{zx} & 0 & \overline{I} \\ -P_{xy} & -\overline{I} & 0 \end{bmatrix}_i \tag{18}$$

$$[C_i'] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I' & P_{zy} \\ 0 & P_{zy} & -I'' \end{bmatrix}_i \tag{19}$$

$$\begin{aligned} P_{xy} &= \int x\, y\, dm_i \\ P_{yz} &= \int yz\, dm_i \\ P_{xz} &= \int zx\, dm_i \\ I' &= I_x - I_z \\ I'' &= I_x - I_y \\ \bar{I} &= (1/2)\,(I_x - I_y - I_z) \end{aligned} \tag{20}$$

and

$$\vec{T}_i' = [r]_i\, \vec{T}_i \tag{21}$$

Combining Equations (5) and (15) yields

$$[M_i]\, \ddot{\vec{\delta}}_i + 2\omega_o [G_i]\, \dot{\vec{\delta}}_i - \omega_o^2\, [C_i]\, \vec{\delta}_i = \vec{F}_i + \omega_o^2\, \vec{S}_i \tag{22}$$

where $[M_i]$ is the element mass matrix given by

$$[M_i] = \left[\begin{array}{ccc|c} m_i & o & o & \\ o & m_i & o & \text{Null} \\ o & o & m_i & \\ \hline & \text{Null} & & I_i \end{array}\right] \tag{23}$$

$$\vec{\delta}_i = \left\{ \begin{array}{c} \vec{\delta}_i' \\ \hline \vec{\theta}_i \end{array} \right\} \tag{24}$$

$$[G_i] = \left[\begin{array}{c|c} m_i\,\beta & \text{Null} \\ \hline \text{Null} & G_i' \end{array}\right] \tag{25}$$

$$[C_i] = \left[\begin{array}{c|c} -m_i\beta^2 & \text{Null} \\ \hline \text{Null} & C_i' \end{array}\right] \tag{26}$$

$$\vec{F}_i = \left\{\begin{array}{c} \vec{F}_i' \\ \hline \vec{T}_i' \end{array}\right\} \tag{27}$$

and

$$\vec{S}_i = \left\{\begin{array}{c} -m_i\beta^2 \rho_i \\ \hline \vec{P}_i \end{array}\right\} \tag{28}$$

To obtain the rotor (R) equivalent of the housing dynamic equation, Equation (1), we assemble Equation (22) for all nodes to obtain:

$$\begin{aligned} [M_R]\{\ddot{\delta}_R\} + [D_R]\{\dot{\delta}_R\} + 2\omega_o[G]\{\dot{\delta}_R\} \\ + [K_R]\{\delta_R\} - \omega_o^2 [C]\{\delta_R\} \\ = \{F_R\} - \{F_{RB}\} + \omega_o^2\{S\} \end{aligned} \tag{29}$$

where

$[D_R]$ is the rotor damping matrix

$[K_R]$ is the rotor assembled stiffness matrix

$[G] = \sum_i [G_i]$

$[M_R] = \sum_i [M_i]$

$[C] = \sum_i [C_i]$

$\{F_R\}$ are the rotor torques given by

$$\{F_R\} = \sum_i \vec{F}_i + [K_R]\{\delta_R\} + [D_R]\{\dot{\delta}_R\} + \{F_{RB}\}$$

$$\{S\} = \sum_i \vec{S}_i$$

and $\{F_{RB}\}$ are the rotor bearing loads which couple the rotor motions $\{\delta_R\}$ to the housing motions $\{\delta_H\}$.

ROTOR AND HOUSING COUPLING

The primary rotor/housing physical coupling is through the bearings and drive system. These bearings are represented in the present study as radial and axial springs through the NASTRAN CELAS computer subroutine element. The motor/servo drive coupling uses a CELAS element for a torsion spring as well.

The primary sources of loads are associated with rotor mass imbalance, bearing imperfections, friction and motor torque ripple.

Since the rotor motion is described in relation to a rotating coordinate system, the mass imbalance terms, $\omega_o^2 \{\vec{S}\}$ in Equation (29), act as static loads upon the rotor. However, these induce bearing loads upon the housing which are periodic (with period $2\pi/\omega_o$) and synchronized with the spin speed (ω_o), thus producing dynamic motion at the rotor bearings.

In addition to these considerations, there will be small bearing imperfections which induce dynamic loads at various ratios, N_i, of the spin speed. These will induce bearing motions at the frequency $N_i\omega_o$ with respect to the housing and $|N_i-1|\omega_o$ with respect to the rotor. The ratios N_i may be calculated from established formulas which depend upon the bearing geometry (ball size, race diameters, etc.), and the magnitude of these disturbances at the bearings must be estimated or determined by test. In a similar manner, motor ripple will appear to have frequencies $M_i\omega_o$ with respect to the housing coordinate system and $|M_i-1|\omega_o$ for the rotor coordinate system.

MSC/NASTRAN MODIFICATIONS

For the purposes of this study, all the rotor and housing elements were modeled as beam elements (CBAR) with finite shear stiffness and rotatory inertia. Flange connections along the shafts were modeled as CELAS elements.

It can be seen from Equation (29) that the rotation has caused three additional types of force terms to arise in the mathematical description for the linearized system. Thus, in addition to the usual symmetric mass, dissipative, stiffness and forcing terms ([M], [D], [K] and {F}, respectively) which NASTRAN usually treats, it is necessary to consider the terms associated with $2\omega_o[G]$, $\omega_o^2[C]$ and $\omega_o^2\{S\}$.

The $2\omega_o[G]$, and $\omega_o^2[C]$ matrices must be individually computed using Equations (18), (19), (25) and (26) and supplied to MSC-NASTRAN as direct input matrices. These matrices can then be automatically constrained, partitioned and assembled in the GKAD Direct Matrix Assembler of the MSC/NASTRAN program[5].

To assist in the generation of these matrices and permit modeling simplifications which are physically justifiable, it is useful to have some understanding and interpretation of the various rotational terms.

The right-hand side term, $\omega_o^2\{\vec{S}\}$, is a forcing term which results from centrifugal action of off-axis, $\vec{\rho}_i$, mass imbalance, or lack of symmetry which produces products of inertias $(P_{xz})_i$ or $(P_{xy})_i$ about the undeflected spin axis. The combination of all such terms are usually referred to as static and dynamic imbalance.

The last term on the left-hand side is caused by the dynamic shift of the original static mass imbalance as well as dynamic shifts caused by deformation of the original spin axis. This matrix, together with $[M_R]$ and $[K_R]$, as will be discussed later, determines the dynamic stability of the spinning system.

The [G] matrix, unlike all the previous mentioned matrices, is skew-symmetric with zeros on the diagonal. This matrix arises from the Coriolis acceleration effects and gives rise to what are often referred to as gyroscopic terms. Its presence couples all rotor motions normal to the spin axis and creates complex natural modes with real and imaginary parts. Fortunately, if one is only concerned with the dynamic stability of a given system this term is only of secondary importance as will be discussed in the following section on stability.

SYSTEM STABILITY

Stability of the dynamic system, described by Equation (29), may be examined by determining the eigenvalues, s_i, of the unsymmetric quadratic eigenvalue problem:

$$(s_i^2 [M] + s_i [D+\overline{G}] + [K-\overline{C}]) \{X^i\} = \{0\} \tag{30}$$

and examining the real portions of all the s_i.

In general, the eigenvalues, s_i, will be complex numbers and the system's stability will be governed by the signs of the real part of the eigenvalues. If any s_i has a positive real part, the entire system is considered unstable and small linear perturbations will grow until the linearized equations are no longer applicable.

If stability of the system is all one is interested in, however, it is not necessary to solve Equation (30). This is due to the Kelvin-Tait-Chetaev Theorem[6] which states that if a system contains both gyroscopic and dissipative forces, then its stability is the same as for the truncated system given by:

$$[K-\overline{C}] \{X^i\} = \lambda_i [M] \{X^i\} \tag{31}$$

where

$$\lambda_i = -s_i^2$$

Since a real rotating system will always contain some dissipative damping, and since the eigenvalues of a real symmetric system must be real, it is only necessary to determine if Equation (31) has a negative λ_i to check for system instability. Since a $\lambda_i < o$ implies at least one real value for s_i, such a system would be unstable. This is generally a much simpler computational task than solving for the eigenvalues of Equation (30) (which is quadratic and unsymmetric). In fact, this simplification is quite useful for MSC/NASTRAN applications since it is not necessary to directly input the [G] matrix, and all of the standard eigenvalue routines included in its library may be used.

LOW SPIN SPEED APPROXIMATIONS

Natural Frequencies

Various approximations for the solution of the governing equations are possible if the spin speed, ω_o, is low with respect to the system's fundamental bending frequency. This statement is based upon the fact that the lowest natural frequencies of a uniform rotating beam, ω_{iR}, are related to the nonrotating frequencies, ω_i, through the relationship

$$\omega_{iR} = \omega_i \left[1 - (\omega_o/\omega_i)^2 \right]^{1/2}, \quad i = 1, 2, \ldots \tag{32}$$

for

$$\omega_o < 1$$

Based upon this result, it seems reasonable to assume that we may solve for the natural frequencies of the combined FEM, including housing and rotor with bearing couplings, by ignoring all rotational terms in the unforced equations. By then comparing the approximate frequency error terms

$$\left[1 - (\omega_o/\omega_i)^2 \right]^{1/2}$$

it is then possible to estimate the validity of the original approximation and to consider these as correction factors.

Modal Reduction

Use can also be made of the unspun natural vibration information to greatly reduce the original dynamic system into fewer degrees of freedom.

This can be effected by using all the lower mode shapes of the unspun system, $\{\phi_u{}^i\}$, in the approximation

$$\left\{\begin{array}{c}\delta_H \\ --- \\ \delta_R\end{array}\right\} \approx \left[\ \Phi\ \right] \left\{\begin{array}{c}u_1 \\ \vdots \\ u_n\end{array}\right\} \tag{33}$$

where the $\{\phi_u{}^i\}$, i = 1, 2,.., n, are the columns of $[\Phi]$ and all the ω_i satisfy

$$\omega_i < 5\omega_o\ ,\ i = 1,\ 2,\ ..,\ n \tag{34}$$

Equation (33) is substituted into Equations (1) and (29) and the results are premultiplied by $[\Phi]^T$ to obtain the reduced rotating system equations. This will generally result in a system which is much smaller to analyze than the original one, and the effects of rotation should be adequately accounted for by virtue of the modest reduction criterion suggested by Equation (34).

Frequency Response

Another approximation which is possible for small ω_o/ω_1, is based on the known exact steady-state solutions to the single degree of freedom equations

$$M\ddot{x}_1 + C\dot{x}_1 + K_1\, x_1 = F \sin\,(\omega_o t + \phi) \tag{35}$$

and

$$M\ddot{x}_2 + (K_1 - K_2)\, x_2 = F \sin\,\omega_o t \tag{36}$$

The steady-state solutions to Equations (35) and (36) are

$$x_i = A_i \sin \omega_o t,\ i = 1,\ 2 \tag{37}$$

where

$$A_1 = \frac{F/K_1}{1-(\omega_o/\omega_1)^2} \quad [1 + \tan^2\phi]^{-1/2} \tag{38}$$

$$A_2 = \frac{F/K_1}{1-(\omega_o/\omega_1)^2} \left[1- \frac{K_2/K_1}{1-(\omega_o/\omega_1)^2}\right]^{-1/2} \tag{39}$$

$\omega_1^2 \equiv K_1/M$

and $\tan\phi = \dfrac{C\omega_o}{1-(\omega_o/\omega_1)^2}$

Since

$$[1 + \tan^2\phi]^{-1/2} = 1 - (1/2)\tan^2\phi + (3/8)\tan^4\phi - \ldots. \tag{40}$$

and

$$[1 - K_2/K_1]^{-1/2} = 1 + \frac{K_2/K_1}{[1-(\omega_o/\omega_1)^2]} + \frac{(K_2/K_1)^2}{[1-(\omega_o/\omega_1)^2]^2} + \ldots \tag{41}$$

it is possible to arrive at the series solution forms of Equations (40) and (41) by replacing Equations (35) and (36) with the iterative forms:

$$M\ddot{x}_1^{(j)} + K_1 x_1^{(j)} = F \sin(\omega_o t + \phi) - C\dot{x}_1^{(j-1)} \tag{42}$$

and

$$M\ddot{x}_2^{(j)} + K_1 x_2^{(j)} = F \sin\omega_o t + K_2 x_2^{(j-1)} \tag{43}$$

and solving for $x_1^{(j)}$ and $x_2^{(j)}$, $j = 1, 2, \ldots.$ starting with $x_1(0) = x_2(0) = 0$.

This suggests replacing Equation (29) with the approximate form:

$$[M_R]\{\ddot{\delta}_R^{(j)}\} + [D_R]\{\dot{\delta}_R^{(j)}\} + [K_R]\{\delta_R^{(j)}\}$$

$$= \{F_R\} - \{F_{RB}\} + \omega_o^2\{S\} \tag{44}$$

$$-2\omega_o[G]\{\dot{\delta}_R j{-}1\} + \omega_o^2[C]\{\delta_R^{(j-1)}\}$$

and solving for the steady-state response $\{\delta_R^{(j)}\}$ in an iterative manner, using the frequency response module in NASTRAN, and starting with $\{\delta_R^{(0)}\} = \{0\}$.

CONCLUSION

Work is currently underway on the SMS using the techniques discussed in this paper and the MSC/NASTRAN program. It is planned that numerical results and experience gained from this effort will be reported upon in the near future.

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