APPLICATION OF SYMMETRY CONDITIONS IN RADIATION EXCHANGE BETWEEN SURFACES

Dr. Mahesh Morjaria
Technical Specialist
XEROX Corporation
800 Phillips Road
Webster, NY 14580

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ABSTRACT

For the analysis of radiation exchange between surfaces, it is not posssible to take symmetry of a model into account in the present version of MSC/NASTRAN. A method which can substantially reduce both the computational and modelling effort by allowing symmetry conditions to be utilized in the analysis is presented. This method is based on the fact that the matrix of influence coefficients, required for the analysis, has a recognizable pattern in the case of symmetric models. This pattern can be exploited to reduce the computational effort. While derivation of the procedure is fairly involved, the final result can be easily implemented in MSC/NASTRAN through DMAP facility. The method is illustrated through a simple example. This approach can be universally applied to a number of other finite element applications.

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INTRODUCTION

Analysis capability of MSC/NASTRAN for thermal radiation exchange between surfaces is often used for models which have symmetry conditions. For example Genberg *et al.** have used this capability to analyse the behavior of illumination system containing diffuse surfaces. That appplication involves a model which has symmetry about a plane. However, in the presently available analysis of radiation exchange, there is no natural way to take symmetry of the finite element model into account. In this paper, a method by which this can be accomplished is presented. The method is based on the recognition that the matrix of view factors, which is required for the analysis, has a "symmetric" pattern for symmetric models. This pattern can be exploited to reduce the computational effort.

**V. Genberg, D. Oinen & S. Fronheiser, 'Diffuse Illumination with MSC/NASTRAN', presented at MSC/NASTRAN Users Conference, March 1983.*

RADIATION EXCHANGE BETWEEN SURFACES*

The governing equation for radiation exchange between surfaces which are divided into finite elements is given as follows:

$$\{ Q_e \} = -[R_e][G]\{ u_g + T_a \}^4 \qquad (1)^{**}$$

where

$$[R_e] = \sigma [A E_\varepsilon - A E_\alpha [A - F [I - E_\alpha]]^{-1} F E_\varepsilon]^{***} \qquad (2)$$

and

$\{ Q_e \}$ net heat flow into an element due to heat radiation

$[R_e]$ radiation element matrix

$[G]$ transformation matrix. It transforms grid point (or nodal) temperatures to element temperature as follows

$$\{ u_e \}^4 = [G]\{ u_g + T_a \}^4$$

$\{ u_e \}$ element temperature vector. Please note that element temperature is assumed to be uniform over each finite element.

$\{ u_g \}$ grid point tempertures

T_a a constant which converts u_g to absolute temperature

σ Stephan-Boltzman constant

$[A]$ diagonal matrix of surface area of elements

$[E_\varepsilon]$ diagonal matrix of emissivity

$[E_\alpha]$ diagonal matrix of absorptivities

**This material is taken from NASTRAN Theoretical Manual, Section 8.3.4. For theoretical aspects of heat radiation please refer to Siegel and Howell, 'Thermal Radiation Heat Transfer'.*

***If { T } is a vector, { T }^{4} is defined as the vector whose components are the fourth power of { T }.*

****All the matrices will be denoted by **bold lettering** and/or by square brackets ([ ]).*

[F] matrix of exchange coefficients whose element, F_{ij} is as follows

$$F_{ij} = \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j \, dA_i \, dA_j}{\pi r_{ij}^2}$$

r_{ij} length of the line connecting two point on elements i and j

θ_i, θ_j angles between the connecting line and the normals to the elements

A_i, A_j surface area of elements i and j

$F_{ij} = A_i f_{ij}$

f_{ij} fraction of power leaving element j which reaches element i (this is commonly refered to as the view factor)

[I] unit diagonal matrix

HALF SYMMETRY CONDITIONS

If the geometry of the model to be analysed is symmetrical about a plane, and the finite element mesh pattern is also symmetrical about this plane, like a reflection in a mirror, then the model will be refered to as a half symmetric model. It can be shown that the matrix of exchange coefficients, [F], under those conditions, has the following form

$$[F] = \begin{bmatrix} [F_{aa}] & [F_{ab}] \\ [F_{ba}] & [F_{bb}] \end{bmatrix} \qquad (3)$$

where

$$[F_{aa}] = [F_{bb}] \text{ and } [F_{ab}] = [F_{ba}] \qquad (4)$$

This assumes that the elements have been numbered in such a way that [F] can be partitioned, as shown above, into four square matrices, each being half the dimension of [F]. This can be readily surmised from the purely geometrical view factor definition of each element of [F], where $F_{ij} = A_i f_{ij}$, and f_{ij} is the fraction of energy leaving surface element i and reaching surface element j. For the sample problem shown in Figure 1, this would mean $F_{12} = F_{56}$, $F_{14} = F_{58}$, $F_{13} = F_{57}$, $F_{17} = F_{53}$, and so on.

This implies that for a model which has a mirror symmetry (half symmetry conditions) about a plane; only half of the matrix need be calculated. This will reduce right away the view factors calculation effort by half. The form of the matrix shown above will be refered to as the **half symmetry form** for rest of the discussions.

Next, it can be shown that the matrix $[R_e]$, defined earlier in Equation 2, is also of the half symmetry form. That is, it can also be partitioned into four square matrices and only two of which need be evaluated. To show this, it will be assumed, quite reasonably, that the material properties, namely emissivity, ε, and absorptivity, α, also exhibit the half symmetry form. For the simple problem shown in Figure 1, this would mean that ε and α are same for element pairs 1 and 5, 2 and 6, 4 and 8, and 3 and 7. (Assuming symmetry about the y axis). However, please note that these properties could be different for each element numbered 1 to 4. The diagonal matrices $[A]$, $[E_\alpha]$ and $[E_\alpha]$ can each be partitioned in the following half symmetry form:

$$[X] = \begin{bmatrix} [X]^a & [0] \\ [0] & [X]^a \end{bmatrix} \qquad (5)$$

where $[X]^a$ is a square diagonal matrix, half the size of $[X]$, and $[0]$ is the null matrix.

In Appendix 1, it is shown that the product of two half symmetry form matrices is also a half symmetry form matrix. It is quite trivial to show that addition or subtraction of two half symmetry form matrices results into a matrix of the same form. Further, in Appendix 2, it is shown that an inverse of a half symmetry form matrix is also a half symmetry form matrix. Since the operations involved in Equation 2 are all on the half symmetry form matrices, it is quite clear that $[R_e]$ is also of the half symmetry form. This implies that only half of the matrix $[R_e]$ need be evaluated.

So far, it has been shown that the matrix $[R_e]$ in equation (2) is of the half symmetry form and hence the computational effort can be reduced by half in its evaluation. Now the symmetry conditions on the "loading", namely the temperature distribution, will be utilized. As mentioned earlier, the finite element approximation considers the temperature to be uniform over each finite element surface (not necessarily the whole surface). It will be assumed that the temperature distribution is also symmetric about the plane of symmetry of the geometry of the model. In the case of the simple problem of Figure 1, this would mean that temperature on elements 1 to 4 is identical with the corresponding temperature of elements 5 to 8. In other words, the element temperature vector, $\{ u_e \}$, can be partitioned in the following way:

$$\{ u_e \}^4 = \begin{bmatrix} \{ u_e \}^a \\ \\ \{ u_e \}^a \end{bmatrix} \tag{6}$$

Let,

$$\{ e \} = [E_\varepsilon] \{ u_e \}^4 \tag{7}$$

It has been shown in Appendix 4 that $\{ e \}$ can be partitioned as follows:

$$\{ e \} = \begin{bmatrix} \{ e \}^a \\ \cdot \\ \{ e \}^a \end{bmatrix} \tag{8}$$

where

$$\{ e \}^a = [E_\varepsilon]^a \{ u_e \}^a \tag{9}$$

Please note that $[E_\varepsilon]$ is only a diagonal matrix.

Let ,

$$\{ f \} = [F] \{ e \} \tag{10}$$

Since [F] is also a half symmetry form matrix, it can be shown that (Appendix 4) :

$$\{ f \} = \begin{bmatrix} \{ f \}^a \\ \\ \{ f \}^a \end{bmatrix} \tag{11}$$

where,

$$\{ f \}^a = [[F_{aa}] + [F_{ab}]] \{ e \}^a \tag{12}$$

From Equation (9),

$$\{ f \}^a = [[F_{aa}] + [F_{ab}]] [E_\varepsilon]^a \{ u_e \}^a \quad (13)$$

Let,

$$[P] = [A - F (I - E_\alpha)] \quad (14)$$

Since [A], [F], [I] and [E_α] are of the half symmetry form, [P] is also of the same form. Hence, [P] can be partitioned as

$$[P] = \begin{bmatrix} [P_{aa}] & [P_{ab}] \\ [P_{ab}] & [P_{aa}] \end{bmatrix} \quad (15)$$

Let,

$$[H] = [P]^{-1} \quad (16)$$

It has been shown in Appendix 2, that an inverse of a half symmetry form matrix is also of the same form. Hence defining,

$$\{ h \} = [H] \{ f \} \quad (17)$$

we get,

$$\{ h \} = \begin{bmatrix} \{ h \}^a \\ \{ h \}^a \end{bmatrix} \quad (18)$$

where

$$\{ h \}^a = [[H_{aa}] + [H_{ab}]] \{ f \}^a \quad (19)$$

It has been shown in Appendix 3, that

$$[[H_{aa}] + [H_{ab}]] = [[P_{aa}] + [P_{ab}]]^{-1} \quad (20)$$

However, from Equations 14 and 15,

$$[P_{aa}] + [P_{ab}] = [A^a - [F_{aa} + F_{ab}] [I - E_\alpha{}^a]] \quad (21)$$

Continuing the process, it can be readily shown from Equations 1, 2 , 6, 13, 20 and 21 that

$$\{ Q_e \} = \begin{bmatrix} \{ Q_e \}^a \\ \\ \{ Q_e \}^a \end{bmatrix} \tag{22}$$

where,

$$\{ Q_e \}^a = - \sigma \Big[A^a E_\varepsilon{}^a - A^a E_\alpha{}^a [A^a - [F_{aa} + F_{ab}] [I - E_\alpha{}^a]]^{-1} [F_{aa} + F_{ab}] [E_\varepsilon]^a \Big] \{ u_e \}^a \tag{23}$$

Please note the following with regards to the above equation:

- o It is half the size of Equation 1
- o The matrix inversion is on half the dimension of the matrix inversion in Equation 1. Since it is a fully populated matrix, as opposed to banded, the reduction in computational effort just for matrix inversion (which is generally the main contributor to the total computation time) will be of the order of cube power of 2 i.e. 8. There will be other savings as well.

QUARTER SYMMETRY CONDITIONS

If the model is also symmetric about another plane, for example about the x axis as well as the y axis in the sample problem shown in Figure 1, the matrix, [F] of Equation (2) can be partitioned into the following quarter symmetry form:

$$[F] = \begin{bmatrix} [F_{aa}] & [F_{ab}] & [F_{ac}] & [F_{ad}] \\ [F_{ab}] & [F_{aa}] & [F_{ad}] & [F_{ac}] \\ [F_{ac}] & [F_{ad}] & [F_{aa}] & [F_{ab}] \\ [F_{ad}] & [F_{ac}] & [F_{ab}] & [F_{aa}] \end{bmatrix} \tag{24}$$

where each square matrix is of the quarter the dimension of [F]. Following the same reasoning as before, it can be easily shown that for quarter symmetry conditions, Equation 23 will reduce to the following form:

$$\{Q_e\}^a = -\sigma \Big[A^a E_\varepsilon{}^a - A^a E_\alpha{}^a [A^a - [F_{aa} + F_{ab} + F_{ac} + F_{ad}]$$

$$[I - E_\alpha{}^a]]^{-1} [F_{aa} + F_{ab} + F_{ac} + F_{ad}] \; [E_\varepsilon]^a \Big] \; \{u_e\}^a$$

.... (25)

Please note that the above equation is of one-fourth the size of Equation (1), and hence the reduction in computational effort will be at least of the order of 4^3 i.e. 64.

SAMPLE PROBLEM

Consider the simple example shown in Figure 1. It consists of an open rectangular box with four surfaces each of which is 10" long and 2" wide. The box is modelled by eight finite elements, each of which is 10" long and 1" wide. The box has symmetry both about the x axis and the y axis. The temperature distribution is assumed to be zero for elements 1,5,3 and 7 and equal to 1 for elements 2,4,6 and 8. Hence, it can be readily observed that the "loading" is also symmetric about the x axis and the y axis. And this model satisfies all the quarter symmetry conditions. The entire model without the symmetry conditions was analysed using MSC/ NASTRAN.

For each of the elements, $\alpha = 1$ and $\varepsilon = 1$. The view factors were calculated by MSC/NASTRAN and are given as follows:

$$[F] = \begin{array}{c} \\ \end{array}$$

	1	2	3	4	5	6	7	8
1	0	a	b	c	0	c	d	e
2	a	0	c	0	c	b	e	d
3	b	c	0	a	d	e	0	c
4	c	0	a	0	e	d	c	b
5	0	c	d	e	0	a	b	c
6	c	b	e	d	a	0	c	0
7	d	e	0	c	b	c	0	a
8	e	d	c	b	c	0	a	0

a = 2.805669, b = 2.068633, c = .8070657, d = 1.536083,

$e = 0.9997281$

Using the quarter symmetry method, given in Equation (25), the following steps are carried out for elements 1 and 2 only,

$$\{ u^e \}^a = \begin{bmatrix} 0^4 \\ 1^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{(temperatures)}$$

$$[E_\alpha]^a = [E_\varepsilon]^a = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

$$[A]^a = \begin{bmatrix} 10.0 & 0.0 \\ 0.0 & 10.0 \end{bmatrix}$$

$$[A]^a [E_\alpha]^a = [A]^a [E_\varepsilon]^a = \begin{bmatrix} 10.0 & 0.0 \\ 0.0 & 10.0 \end{bmatrix}$$

$$[I - E_\alpha{}^a] = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

$$[F_{aa} + F_{ab} + F_{ac} + F_{ad}] = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} + \begin{bmatrix} b & c \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & c \\ c & b \end{bmatrix} + \begin{bmatrix} d & e \\ e & d \end{bmatrix}$$

$$= \begin{bmatrix} g & h \\ h & g \end{bmatrix}$$

where $g = 3.604716$ and $h = 5.419773$

$\sigma = 1.0$

Substituitng the above terms in Equation 25, we get

$$\{Q_e\}^a = \begin{Bmatrix} 5.420 \\ -6.395 \end{Bmatrix}$$

The above values agree with the MSC/NASTRAN results.

CONCLUSIONS

A method which will substantially reduce both the computational and modelling effort for the analysis of radiation exchange between surfaces by taking symmetry of a model into account has been presented. The method can be implemented in MSC/NASTRAN through DMAP facility. A sample example illustrating the method has been solved. This technique is equally applicable to other finite element applications where the matrix of influence coefficients (or stiffness matrix) can be partitioned in a symmetric form.

APPENDICES

1. MULTIPICATION OF TWO HALF-SYMMETRY FORM MATRICES

Let [C] = [B] [A] , where [A] and [B] are two half symmetry form matrices. To show that [C] is also of the same form.

$$\begin{bmatrix} [C_{aa}] & [C_{ab}] \\ [C_{ba}] & [C_{bb}] \end{bmatrix} = \begin{bmatrix} [B_{aa}] & [B_{ab}] \\ [B_{ab}] & [B_{aa}] \end{bmatrix} \begin{bmatrix} [A_{aa}] & [A_{ab}] \\ [A_{ab}] & [A_{aa}] \end{bmatrix}$$

$$[C_{aa}] = [B_{aa}][A_{aa}] + [B_{ab}][A_{ab}]$$
$$[C_{ab}] = [B_{aa}][A_{ab}] + [B_{ab}][A_{aa}]$$
$$[C_{ba}] = [B_{ab}][A_{aa}] + [B_{aa}][A_{ab}]$$
$$[C_{bb}] = [B_{ab}][A_{ab}] + [B_{aa}][A_{aa}]$$

Therefore,

$$[C_{aa}] = [C_{bb}]$$
$$[C_{ab}] = [C_{ba}]$$

Q.E.D.

2. INVERSE OF A HALF SYMMETRY FORM MATRIX

Let $[B] = [A]^{-1}$, where [A] is a half symmetry form matrix. To show that [B] is also a half symmetry form matrix. By definition,

$$[A][B] = [I]$$

where [I] is a unit matrix. Or on partition,

$$\begin{bmatrix} [A_{aa}] & [A_{ab}] \\ [A_{ab}] & [A_{aa}] \end{bmatrix} \begin{bmatrix} [B_{aa}] & [B_{ab}] \\ [B_{ba}] & [B_{bb}] \end{bmatrix} = \begin{bmatrix} [I] & [0] \\ [0] & [I] \end{bmatrix}$$

Therefore,

$$[A_{aa}][B_{aa}] + [A_{ab}][B_{ba}] = [I] \qquad (1)$$

$$[A_{aa}][B_{ab}] + [A_{ab}][B_{bb}] = [0] \qquad (2)$$

$$[A_{ab}][B_{aa}] + [A_{aa}][B_{ba}] = [0] \qquad (3)$$

$$[A_{ab}][B_{ab}] + [A_{aa}][B_{bb}] = [I] \qquad (4)$$

From (1) and (4),

$$[B_{aa}] = [A_{aa}]^{-1}[[I] - [A_{ab}][B_{ba}]] \qquad (5)$$

$$[B_{bb}] = [A_{aa}]^{-1}[[I] - [A_{ab}][B_{ab}]] \qquad (6)$$

From equations (3) and (5),

$$[B_{ba}] = [[A_{ab}][A_{aa}]^{-1}[A_{ab}] - [A_{aa}]]^{-1}[A_{ab}][A_{aa}]^{-1} \qquad (7)$$

Similarly, from (2) and (6) ,

$$[B_{ab}] = [[A_{ab}][A_{aa}]^{-1}[A_{ab}] - [A_{aa}]]^{-1}[A_{ab}][A_{aa}]^{-1} \qquad (8)$$

Therefore,

$$[B_{ab}] = [B_{ba}] \qquad (9)$$

Further, from equations (5) and (6),

$$[B_{aa}] = [B_{bb}] \qquad (10)$$

Q.E.D.

3 MATRIX INVERSION

Given,

$$[B] = [A]^{-1}$$

where [A] is a half symmetry form matrix, to show

$$[[B_{aa}] + [B_{ab}]] = [[A_{aa}] + [A_{ab}]]^{-1}$$

Adding equations (1) and (2) and using the conditions (9) and (10) of Appendix 2, we get

$$[A_{aa}][B_{aa}] + [A_{ab}][B_{ab}] + [A_{aa}][B_{ab}] + [A_{ab}][B_{aa}] = [I]$$

Therefore,

$$[[A_{aa}] + [A_{ab}]]\,[[B_{aa}] + [B_{ab}]] = [I]$$

And hence,

$$[B_{aa}] + [B_{ab}] = [[A_{aa}] + [A_{ab}]]^{-1}$$

Q.E.D.

4 PRODUCT OF A HALF SYMMETRIC FORM MATRIX AND A VECTOR

Given, [A] is a half symmetry form matrix and

$$\{x\} = \begin{bmatrix} \{x\}^a \\ \{x\}^a \end{bmatrix}$$

To obtain $\{y\} = [A]\{x\}$ i.e. on partitioning,

$$\begin{bmatrix} \{y\}^a \\ \{y\}^b \end{bmatrix} = \begin{bmatrix} [A_{aa}] & [A_{ab}] \\ [A_{ab}] & [A_{aa}] \end{bmatrix} \begin{bmatrix} \{x\}^a \\ \{x\}^a \end{bmatrix}$$

Therefore, it can be readily seen that

$$\{y\}^a = \{y\}^b = [[A_{aa}] + [A_{ab}]]\{x\}^a$$

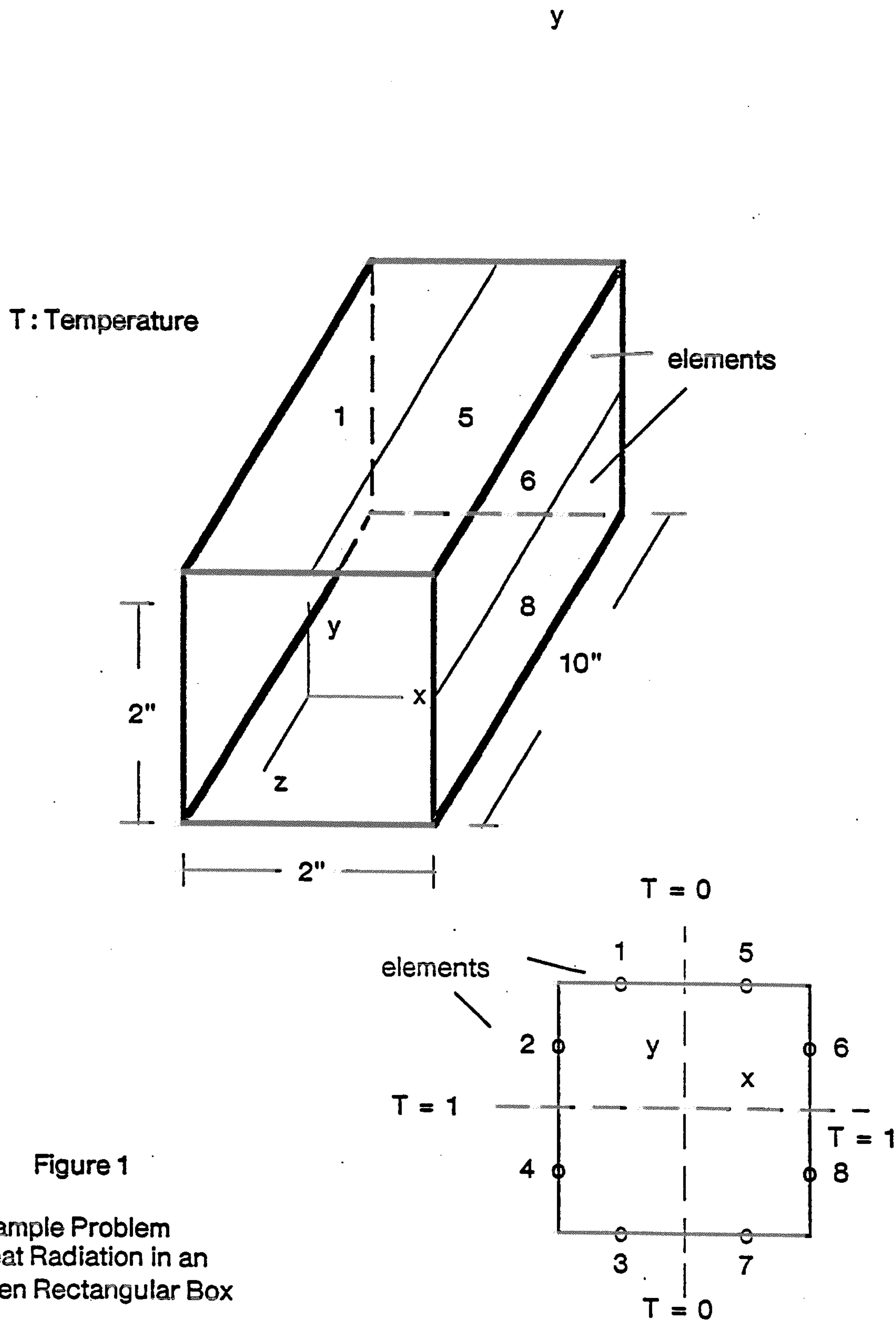

Figure 1

Sample Problem
Heat Radiation in an
Open Rectangular Box