A MODERN FAMILY OF CRACK TIP ELEMENTS

FOR MSC/NASTRAN

by

Jatin C. Parekh
Rocky R. Arnold
Peter J. Woytowitz

Anamet Laboratories, Inc.
San Carlos, California 94070

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J. C. Parekh, R. R. Arnold, and P. J. Woytowitz
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ABSTRACT

Two new Crack Tip Elements (CRAC2D and CRAC3D) have been developed and incorporated into MSC/NASTRAN. The elements are considered linear, isotropic, and homogeneous. Mode I, II, and III stress intensity factors are automatically calculated. Comparisons to theoretical solutions for several geometries are presented and demonstrate the accuracy of the developed elements. Extensions of the elements are discussed.

1.0 INTRODUCTION

Crack or singular elements have been developed for finite element analysis for almost as long as finite element codes have been available. These elements usually are classified as either hybrid or singular element formulations. Many of the elements developed suffered from either lack of accuracy, generality, or consistency. Barsoum [1] points out shortcomings of several different elements. These shortcomings include inability to model rigid body or constant strain modes, inability to include thermal or body force effects, and lack of compatibility with other elements.

The elements developed by Barsoum [1] and Henshell [2] rectified many of the problems described above; however, these elements were limited to displacement of the form $r^{1/2}$. Consequently, they could only model strain singularities of the form $r^{-1/2}$. Recently, Stern [3] and more recently, Hughes and

Akin [4] have introduced families of consistent, conforming elements which allow displacements of the form r^γ. While the Stern element appears to have the restriction that $0 < \gamma < 1$, the element of Hughes and Akin is valid for all $\gamma > 0$.

A 2-D triangular element based upon the shape functions of Hughes and Akin [4] was developed and incorporated into COSMIC/NASTRAN by Woytowitz and Citerley [5]. This triangular element was selected as the basis for the 2-D element (CRAC2D) presented here. The 3-D version (CRAC3D) is a new development and is based upon an analogous extension of the 2-D theory.

The elements presented here possess the required rigid body and constant strain modes. They properly model thermal and gravity loading conditions. Additionally, they are compatible with standard linear or quadratic isoparametric elements.

Figure 1 depicts the quadrilateral and symmetric half options for the CRAC2D element construction. The element is constructed from basic triangluar elements labeled 1 through 8. Figure 2 shows how the 3-D brick and symmetric half crack options for the CRAC3D element are constructed. The brick element is constructed from basic wedge elements labeled 1 through 8, whereas the symmetric half crack option is constructed from basic wedges 1 through 4.

The CRAC2D element is based upon a 2-D formulation, but may be used in three dimensional structures. However, the element should be planar. Any deviation from a planar element is checked, and if significant deviations arise, error messages will be issued.

The CRAC3D element is based upon a 3-D formulation. Both the faces (formed by grids 1-18 and grids 19-36) and the midplane (grids 37-46) should be planar. Any significant deviation will be checked and error messages will be issued.

2.0 THEORY AND FORMULATION OF THE CRACK TIP ELEMENTS

2.1 SHAPE FUNCTIONS FOR THE CRAC2D CRACK TIP ELEMENT

The CRAC2D element is used to model the singular stress and strain fields which exist near the sharp edge of a crack in a 2-D continuum. The element is formed by assembling either four or eight basic triangular elements together. The geometry and displacements inside each triangular element are represented in terms of shape functions. These shape functions were first presented by Hughes and Akin [4] and the following derivation follows their work. The basic triangular element is to be formed by degenerating an eight node isoparametric element.

Referring to Figure 3, the standard bilinear shape functions are used for grids 1 through 4:

$$\begin{aligned} N_1(r,s) &= (1-r)(1-s) \\ N_2(r,s) &= r(1-s) \\ N_3(r,s) &= rs \\ N_4(r,s) &= (1-r)s \end{aligned} \tag{1}$$

The preliminary shape functions for grids 5 - 8 are chosen as:

$$\begin{aligned} N_5(r,s) &= (1-s)P(r,\gamma) \\ N_6(r,s) &= rP(s,2) \\ N_7(r,s) &= sP(r,\gamma) \\ N_8(r,s) &= (1-r)P(s,2) \end{aligned} \tag{2}$$

where

$$P(x,\gamma) = 2\left(x - \frac{x^{\gamma} - 2(1/2)^{\gamma}x}{1 - 2(1/2)^{\gamma}}\right) \tag{3}$$

In order for the shape functions associated with grids 1 - 4 to satisfy the interpolation property at grids 5 - 8, the shape functions for grids 1 - 4 are modified as follows:

$$
\begin{aligned}
N_1 &\leftarrow N_1(r,s) - [N_8(r,s) + N_5(r,s)]/2 \\
N_2 &\leftarrow N_2(r,s) - [N_5(r,s) + N_6(r,s)]/2 \\
N_3 &\leftarrow N_3(r,s) - [N_6(r,s) + N_7(r,s)]/2 \\
N_4 &\leftarrow N_4(r,s) - [N_7(r,s) + N_8(r,s)]/2
\end{aligned} \tag{4}
$$

where the $\leftarrow$ reads: "is replaced by".

The shape functions for all eight grids now satisfy the required interpolation property. Additionally, the shape functions are capable of exactly representing the monomials 1, r, s, r^γ, rs, s^2, $r^\gamma s$, and $s^2 r$. The presence of 1, r, and s ensure representation of rigid body and constant strain modes. The presence of r^γ allows exact representation of displacements of the form r^γ. Note that this will result in a line singularity of the form $r^{\gamma-1}$ upon differentiation.

In order to represent point singularities, the quadrilateral must be degenerated into a triangle. This is done by coalescing grids 4, 8, and 1 as can be done for standard isoparametric elements and is shown schematically in Figure 4. Thus, finally, for a point singularity, the shape function associated with grid 1 is replaced with:

$$
N_1(r,s) \leftarrow N_1(r,s) + N_4(r,s) + N_8(r,s) \tag{5}
$$

In summary, for the basic triangle, the shape function associated with grid 1 is given by Equation (5), the shape functions associated with grids 2 and 3 are given by N_2 and N_3 of Equation (4), and the shape functions associated with grids 5 through 7 are given by N_5 through N_7 of Equation (2).

2.2 SHAPE FUNCTIONS FOR THE CRAC3D CRACK TIP ELEMENT

The CRAC3D element is used to model the singular stress and strain fields which exist near the sharp edge of a crack in a 3-D continuum. The element is formed by assembling either four or eight wedge elements together. The geometry and displacements inside each wedge element are represented in terms of shape functions which were first presented by Hughes and Akin [4].

Referring to Figure 5, the preliminary shape functions for the corner grids of the wedge are selected as

$$N_1(r,s,t) = \tilde{N}_1(r,s)(t+1)/2$$
$$N_2(r,s,t) = \tilde{N}_2(r,s)(t+1)/2$$
$$N_3(r,s,t) = \tilde{N}_3(r,s)(t+1)/2$$
$$N_4(r,s,t) = \tilde{N}_1(r,s)(1-t)/2$$
$$N_5(r,s,t) = \tilde{N}_2(r,s)(1-t)/2$$
$$N_6(r,s,t) = \tilde{N}_3(r,s)(1-t)/2 \qquad (6)$$

Here the $\tilde{N}_i$'s are the 2-D shape functions presented in Section 2.1. The shape functions for the mid-side grids are chosen as

$$N_7\,(r,s,t) = \tilde{N}_5(r,s)(t+1)/2$$
$$N_8\,(r,s,t) = \tilde{N}_6(r,s)(t+1)/2$$
$$N_9\,(r,s,t) = \tilde{N}_7(r,s)(t+1)/2$$
$$N_{10}(r,s,t) = \tilde{N}_5(r,s)(1-t)/2$$
$$N_{11}(r,s,t) = \tilde{N}_6(r,s)(1-t)/2$$
$$N_{12}(r,s,t) = \tilde{N}_7(r,s)(1-t)/2$$
$$N_{13}(r,s,t) = \tilde{N}_1(r,s)(1-t^2)$$
$$N_{14}(r,s,t) = \tilde{N}_2(r,s)(1-t^2)$$
$$N_{15}(r,s,t) = \tilde{N}_3(r,s)(1-t^2) \qquad (7)$$

Next, the shape functions for the corner grids are corrected so that they satisfy the interpolation property at the mid-side grids.

$$N_1(r,s,t) \leftarrow N_1(r,s,t) - \frac{1}{2} N_{13}(r,s,t)$$
$$N_2(r,s,t) \leftarrow N_2(r,s,t) - \frac{1}{2} N_{14}(r,s,t)$$
$$N_3(r,s,t) \leftarrow N_3(r,s,t) - \frac{1}{2} N_{15}(r,s,t)$$
$$N_4(r,s,t) \leftarrow N_4(r,s,t) - \frac{1}{2} N_{13}(r,s,t)$$
$$N_5(r,s,t) \leftarrow N_5(r,s,t) - \frac{1}{2} N_{14}(r,s,t)$$
$$N_6(r,s,t) \leftarrow N_6(r,s,t) - \frac{1}{2} N_{15}(r,s,t) \qquad (8)$$

Now all shape functions satisfy the required interpolation property. To form a linear-edge type wedge, the same shape functions as above are used except that

$$N_{13}(r,s,t) = N_{14}(r,s,t) = N_{15}(r,s,t) = 0. \qquad (9)$$

2.3 GENERATION OF STIFFNESS AND MASS MATRICES

As previously described, the CRAC2D and CRAC3D elements are formed by assembling four or eight basic triangular (or wedge) elements and condensing out all interior degrees of freedom except for the center grids. The stiffness and mass matrices for the basic elements are developed following the standard procedures described by Zienkiewicz [6]. These quantities are given in terms of element coordinates as

$$\underset{\sim}{K}^e = \int_{V^e} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, dV$$

$$\underset{\sim}{M}^e = \int_{V^e} \rho \, \underset{\sim}{N}^T \underset{\sim}{N} \, dV$$

$$\underset{\sim}{P}^e = \int_{V^e} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{\alpha} \, \Delta T dV \qquad (10)$$

For the 2-D elements

$$B = \underset{\sim}{L}\,\underset{\sim}{N}\,, \qquad \underset{\sim}{N} = [N_1\,\underset{\sim}{I},\ N_2\,\underset{\sim}{I},\ \ldots]\,, \qquad \underset{\sim}{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \qquad \underset{\sim}{\alpha} = \begin{bmatrix} \alpha \\ \alpha \\ 0 \end{bmatrix}$$

and

$$\underset{\sim}{D} = \begin{bmatrix} (\lambda+2\mu) & \lambda & 0 \\ \lambda & (\lambda+2\mu) & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{11}$$

This D matrix is for plane strain. For plane stress, λ is replaced by $2\lambda\mu/(\lambda+2\mu)$.

The integrations are performed using Gaussian quadrature. That is, the integrals are approximated as:

$$\int f(x)\,dx \simeq \sum_{j=1}^{n} W_j\, f(a_j) \tag{12}$$

For the 3-D elements

$$B = \underset{\sim}{L}\,\underset{\sim}{N}\,, \qquad \underset{\sim}{N} = [N_1\,\underset{\sim}{I},\ N_2\,\underset{\sim}{I},\ \ldots]\,, \qquad \underset{\sim}{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

$$\underset{\sim}{\alpha} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\underset{\sim}{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \text{SYM.} & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \tag{13}$$

and the integrals are approximated as:

$$\int f(x,y,z)\ dV = \sum_{i=1}^{n_r} \sum_{j=1}^{n_s} \sum_{k=1}^{n_t} W_i W_j W_k\ f(r_i, s_j, t_k) \tag{14}$$

The basic element's matrices are then assembled to form the matrices for the CRAC2D (or CRAC3D) elements. The assembly process is depicted in Figure 6 for the CRAC2D element and Figure 7 for the CRAC3D element. To develop the final CRAC2D and CRAC3D matrices, the force-displacement relationship for the element is written as

$$\underset{\sim}{K}\,\underset{\sim}{U} = \underset{\sim}{F} \quad \text{or} \quad \begin{bmatrix} \underset{\sim}{K}_{rr} & \underset{\sim}{K}_{ro} \\ \underset{\sim}{K}_{or} & \underset{\sim}{K}_{oo} \end{bmatrix} \begin{bmatrix} \underset{\sim}{U}_{r} \\ \underset{\sim}{U}_{o} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{F}_{r} \\ \underset{\sim}{F}_{o} \end{bmatrix} \tag{15}$$

where the o subscript denotes the degrees of freedom to be omitted while the r subscript denotes the degrees of freedom to be retained. It may be assumed here that no external forces are applied to the internal degrees of freedom so that $F_o = 0$ in Equation (15). The relationship between the displacements may then be written as

$$\begin{bmatrix} \underset{\sim}{U}_{r} \\ \underset{\sim}{U}_{o} \end{bmatrix} = \underset{\sim}{T}\,\underset{\sim}{U}_{r} \tag{16}$$

where

$$\underset{\sim}{T} = \begin{bmatrix} I_{rr} & \\ -K_{oo}^{-1} & K_{or} \end{bmatrix}$$

The CRAC2D (or CRAC3D) stiffness and mass matrices are then given as

$$K^* = \underset{\sim}{T}^T\,\underset{\sim}{K}\,\underset{\sim}{T}$$

$$M^* = \underset{\sim}{T}^T\,\underset{\sim}{M}\,\underset{\sim}{T} \tag{17}$$

where K and M are the unreduced stiffness matrices including the internal degrees of freedom and K* and M* are the final reduced CRAC2D (or CRAC3D) matrices.

The thermal load vectors are developed in a completely analogous manner to the stiffness and mass matrices. The thermal loads for each basic element are computed and assembled to form the unreduced thermal load vector. By equating the unreduced thermal load vector to the right hand side of Equation (15), the CRAC2D (or CRAC3D) thermal load vector is then given as the following reduced thermal load vector

$$P^* = \underset{\sim}{T}^T \underset{\sim}{P} \tag{18}$$

where

$$\underset{\sim}{P} = \begin{bmatrix} \underset{\sim}{P}_r \\ \underset{\sim}{P}_o \end{bmatrix}$$

is the unreduced thermal load vector including the internal degrees of freedom and the matrix T is defined by Equation (16).

2.4 ELEMENT STRESS RECOVERY

Stress recovery for the CRAC2D and CRAC3D elements consists of reporting both representative stress values and the values of the stress intensity factors for mode I (K_I), mode II (K_{II}), and mode III (K_{III}) fracture. In order to compute quantities for any basic element, the displacements of the interior degrees of freedom which were condensed out must first be recovered. These displacements are given by

$$\underset{\sim}{U}_o = - \underset{\sim}{K}_{oo}^{-1} \underset{\sim}{K}_{or} \underset{\sim}{U}_r + \underset{\sim}{K}_{oo}^{-1} \underset{\sim}{P}_o = \underset{\sim}{T}_{or} \underset{\sim}{U}_r + K_{oo}^{-1} \underset{\sim}{P}_o \tag{19}$$

where the subscripts o and r represent omitted (condensed out) degrees of freedom and the retained degrees of freedom, respectively.

With the appropriate displacements, the stresses and stress intensity factors are then calculated for selected basic elements of the CRAC2D (or CRAC3D) element. Stresses are computed for basic elements 4 and 5. They are then averaged and reported. In addition to the stresses, the coordinates of the point at which they are computed are also calculated and reported.

Stresses are calculated at the natural coordinate centroids. These centroids correspond to the locations of s = r = 1/2 and t = 0 in Figures 4 or 5. The stresses are calculated using the equations

$$\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon} = \underset{\sim}{D}\,\underset{\sim}{B}\,\underset{\sim}{u}^e \tag{20}$$

where $\underset{\sim}{u}^e$ are the element's grid displacements, $\underset{\sim}{D}$ and $\underset{\sim}{B}$ were defined in Section 2.3.

In addition to the stresses, the element coordinates of these stress locations are also calculated. These locations are measured in element coordinates. The x, y, and z locations are given by:

$$x = \sum_i N_i\, x_i^e$$

$$y = \sum_i N_i\, y_i^e$$

$$z = \sum_i N_i\, z_i^e \tag{21}$$

where x_i^e, y_i^e, and z_i^e are the coordinates of the element's grid points measured in element coordinates and the summation is carried out over the number of grid points. The shape functions, N_i, are evaluated at s = r = 1/2 and t = 0.

The stress intensity factors are calculated using displacements. This consistently yields the most accurate results in tests performed to date [7]. The stress intensity factors are given by the equations:

$$K_I = \frac{\mu}{2(1-\nu)} \lim_{r \to 0} \left(\frac{2\pi}{r}\right)^{1/2} u_y \quad (\phi=\pi)$$

$$K_{II} = \frac{\mu}{2(1-\nu)} \lim_{r \to 0} \left(\frac{2\pi}{r}\right)^{1/2} u_x \quad (\phi=\pi)$$

$$K_{III} = \frac{E}{2(1+\nu)} \lim_{r \to 0} \left(\frac{\pi}{2r}\right)^{1/2} u_z (\phi=\pi) \qquad (22)$$

The above nomenclature is depicted in Figure 8. The expressions on the right hand side of the limit signs are extrapolated to $r = 0$ using Lagrangian interpolation. For the CRAC2D element, neither the z coordinate nor the mode III stress intensity are calculated.

3.0 NUMERICAL RESULTS AND ACCURACY OF THE ELEMENTS

Various simple element tests were first performed on the element to ensure proper coding. Figures 9 and 10 show two test problems which were run on the CRAC2D problem. Comparisons to theory are shown in Table 1. The exact solutions were obtained from [8]. Preliminary comparisons of the 2-D element's accuracy were made by Woytowitz and Citerley [5]. Refer to [5] for more details concerning these results.

A sample test problem for the CRAC3D element is a model of the double cantilever beam specimen as shown in Figure 11. The theoretical solution [9] for the tension load case is $K_I = 1{,}815$ lb-in.$^{-3/2}$. The MSC/NASTRAN solution for this case is $K_I = 1{,}876$ lb-in.$^{-3/2}$.

4.0 CONCLUSIONS AND FUTURE RESEARCH

The CRAC2D and CRAC3D elements provide the MSC/NASTRAN User with new powerful tools for ascertaining the damage tolerance, fatigue life, and fracture tolerance of complex structures. The elements have proved to be accurate enough for most engineering work since scatter in critical (K_{Ic}) values alone are typically 10%. However, reasonable margins of saftey must be used for design using this element. The errors will undoubtedly reduce when a finer mesh is used, although it has been found that too fine a mesh tends to increase the error [5]. Studies should be made to ascertain the optimum crack element size as a function of crack length and other geometrical factors. For now, it is recommended that the user try several element sizes and observe the changes in K_I, K_{II}, etc. However, very small element sizes (say, 5% of the crack length) should be avoided. Future research should include determining this optimum element size and expressing it as a function of crack length and other geometrical parameters.

Both elements are easily extendible to anisotropic and nonlinear materials. Additionally, nonlinear geometry effects could also be included. Inclusion of J-integral calculations is also an easy extension and should be implemented. Finally, additional comparisons with both theoretical and experimental solutions, which could lead to possible improvements in the elements, should be made.

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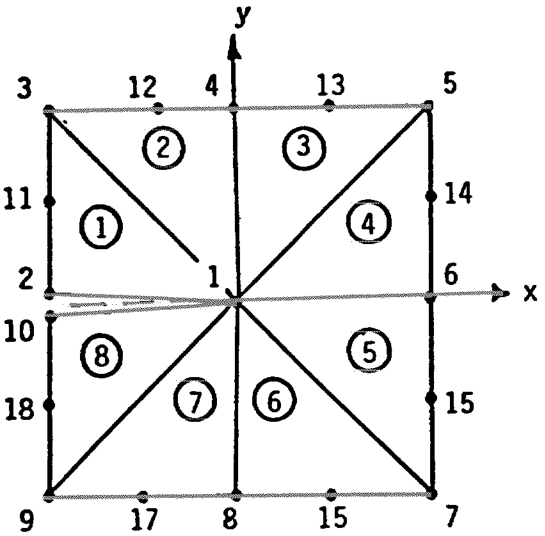
TABLE 1

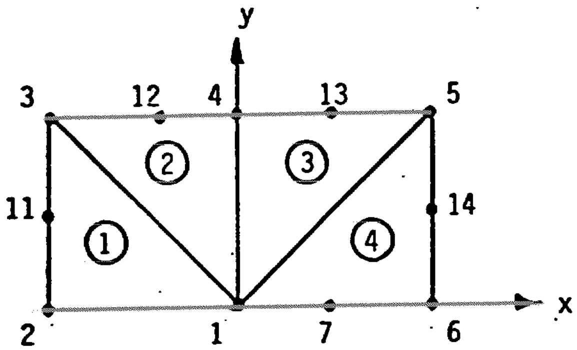
ERRORS IN K_I, K_{II}, AND COD CRAC2D ERRORS

COMPARED TO THEORETICAL RESULTS

	COD	K_I	K_{II}
Edge Crack with Uniform Stress (37 Grid Mesh)	-5.22	-5.43	-
Central Crack in Finite Plate (234 Grid Mesh)	-	2.44	4.48

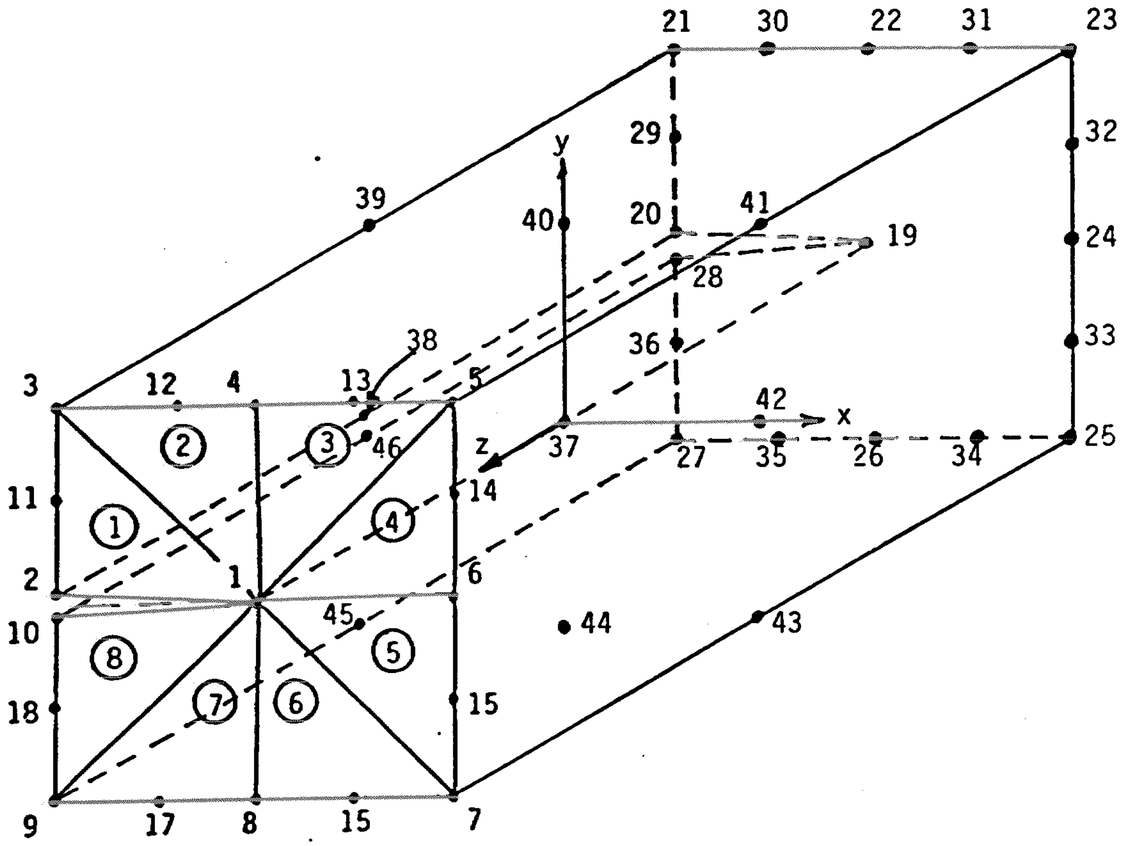


(a) quadrilateral crack element

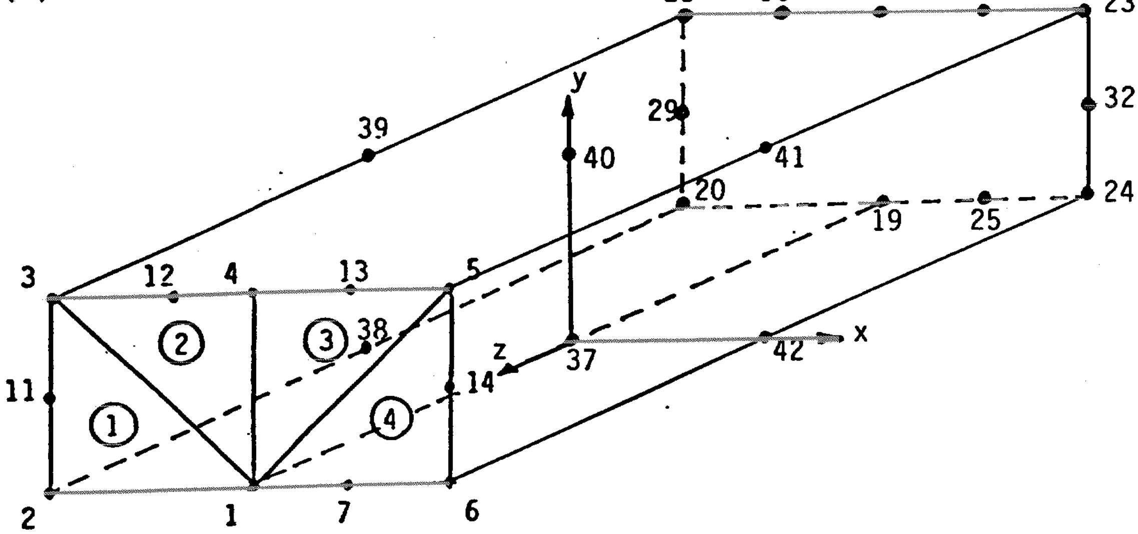


(b) symmetric half-crack option

Figure 1. Geometry Nomenclature for the CRAC2D Element.



(a) brick crack element



(b) symmetric half-crack ootion

Figure 2. Geometry Nomenclature for the CRAC3D Element.

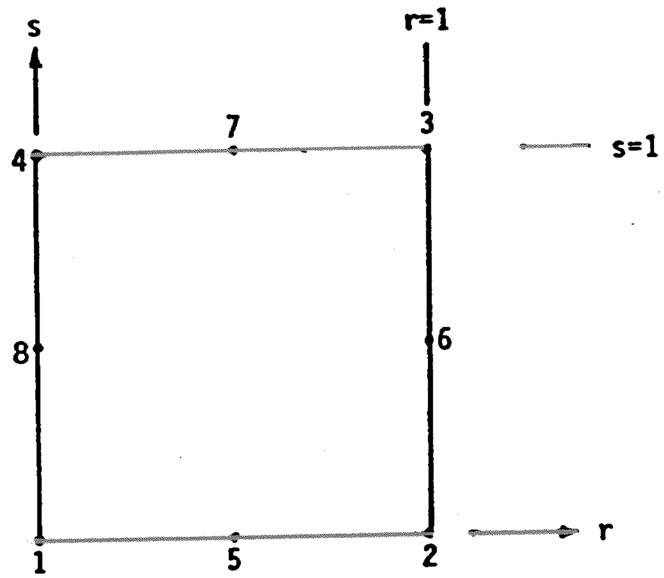


Figure 3. Nomenclature for Eight-Node Isoparametric Element.

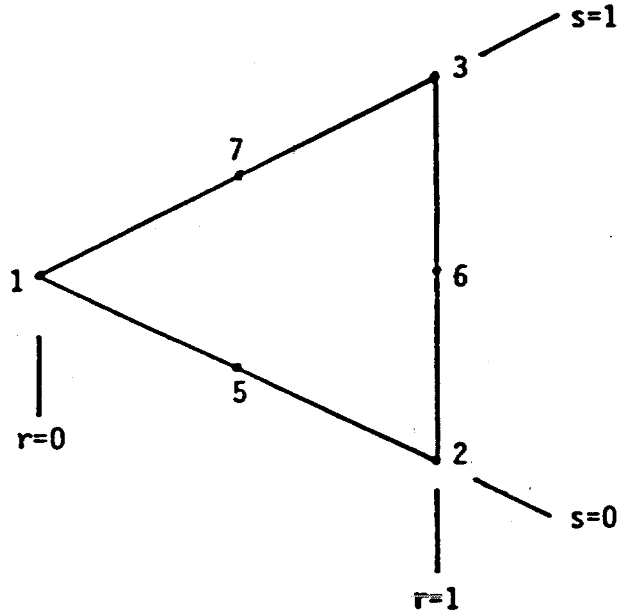


Figure 4. Degeneration of the Eight-Node Element to a Six-Node Triangular Element.

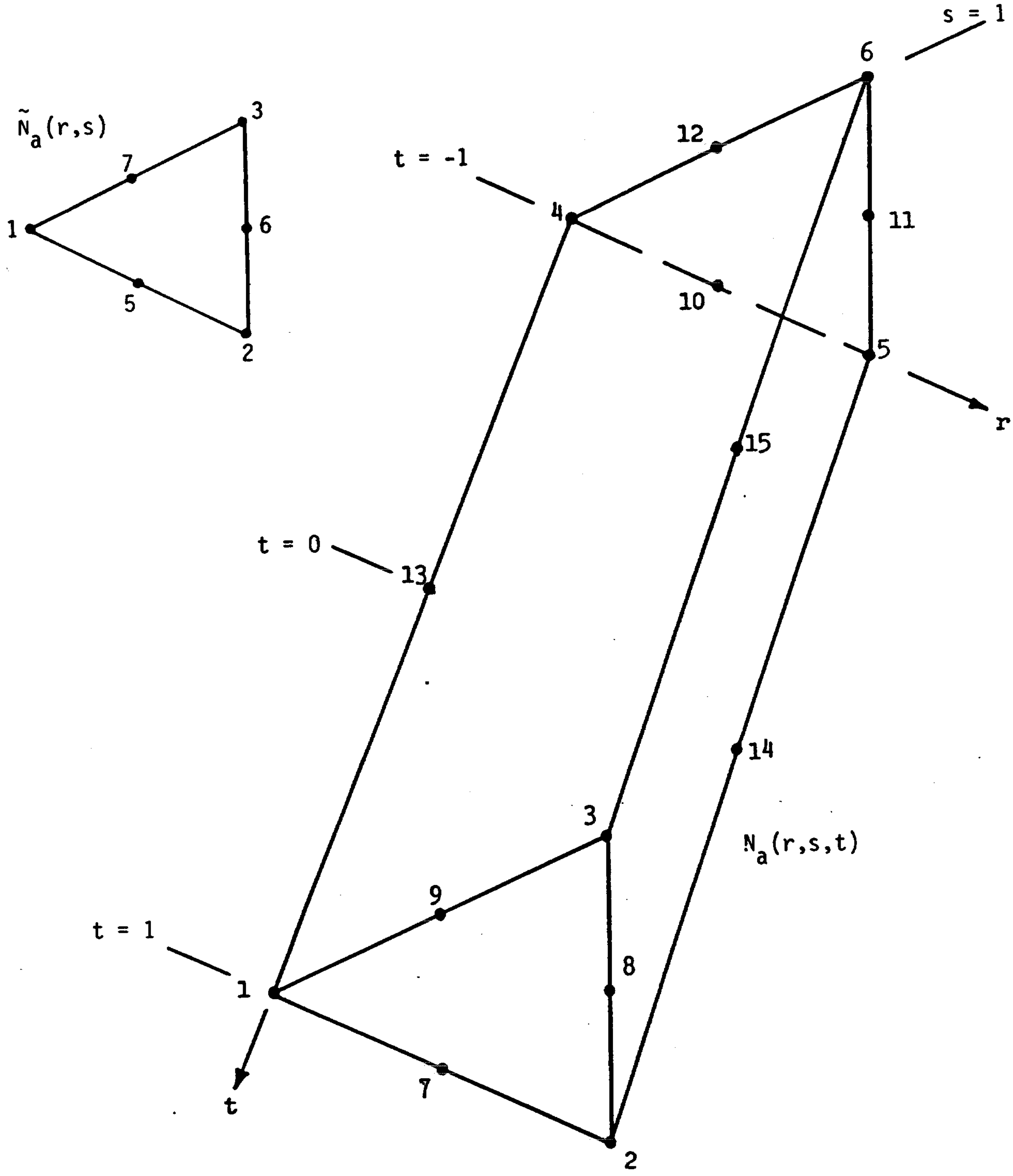


Figure 5. Fifteen-grid 3-D Element.

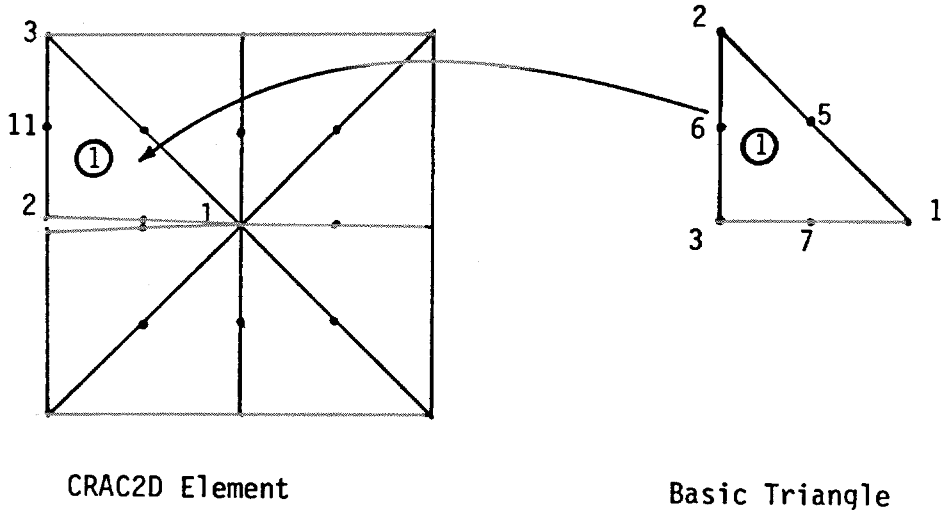
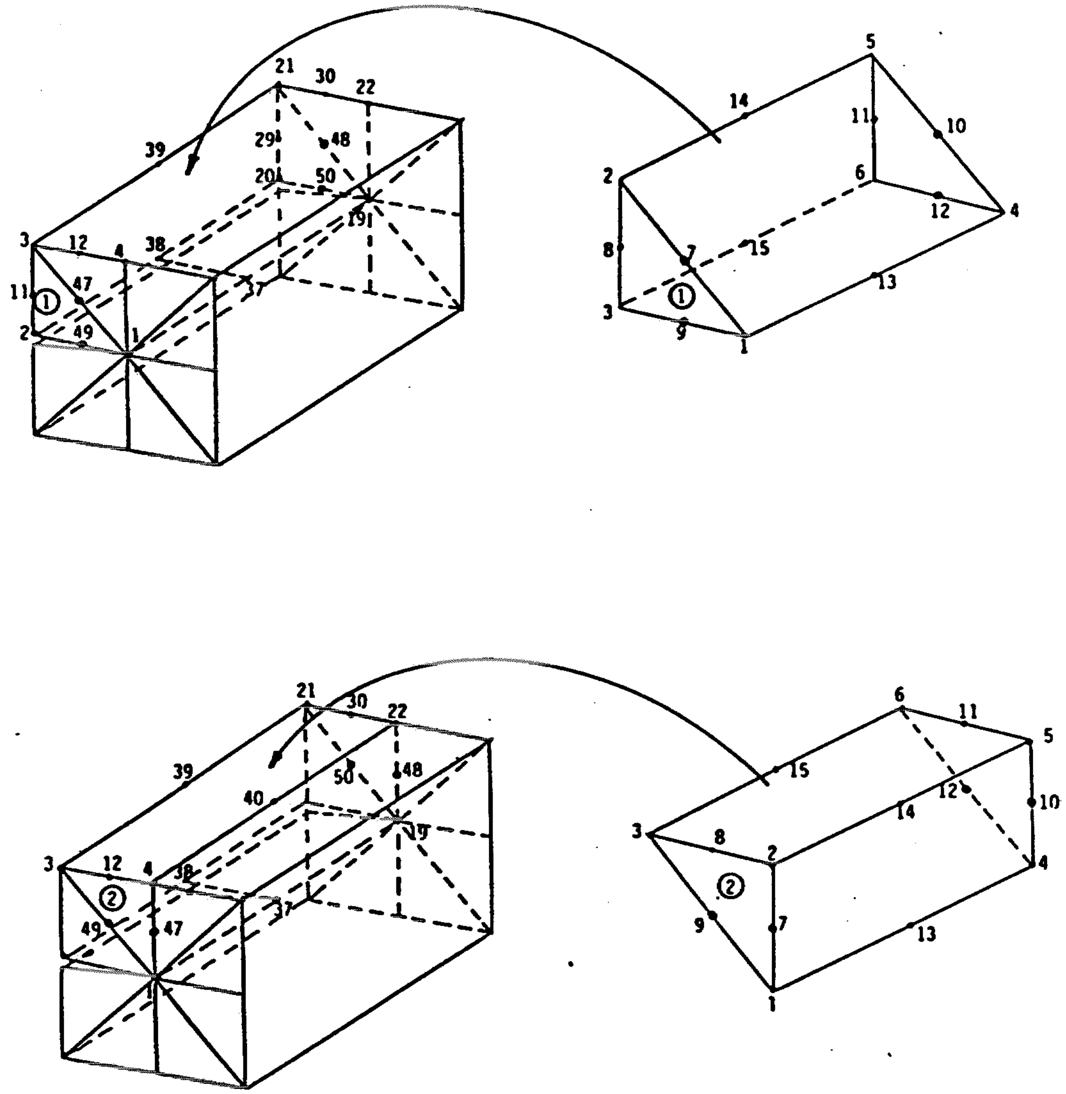


Figure 6. Assembly of Basic Triangles to Form CRAC2D Element.



(repeat for each wedge)

Figure 7. Assembly of Wedge Elements into CRAC3D Element.

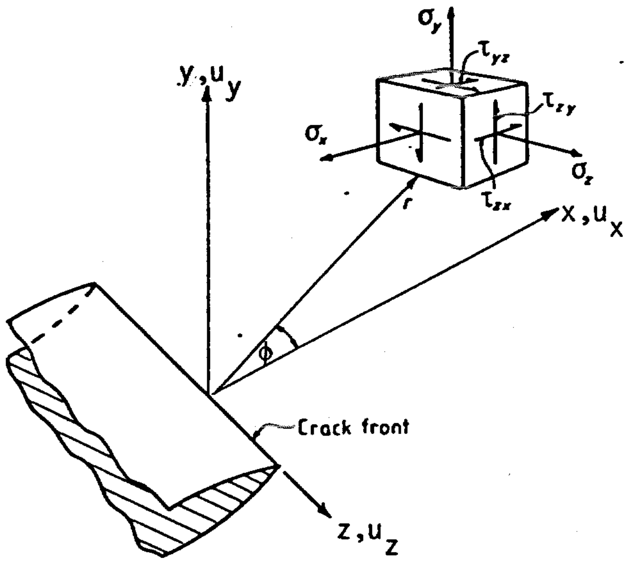


Figure 8. Nomenclature for Crack Geometry.

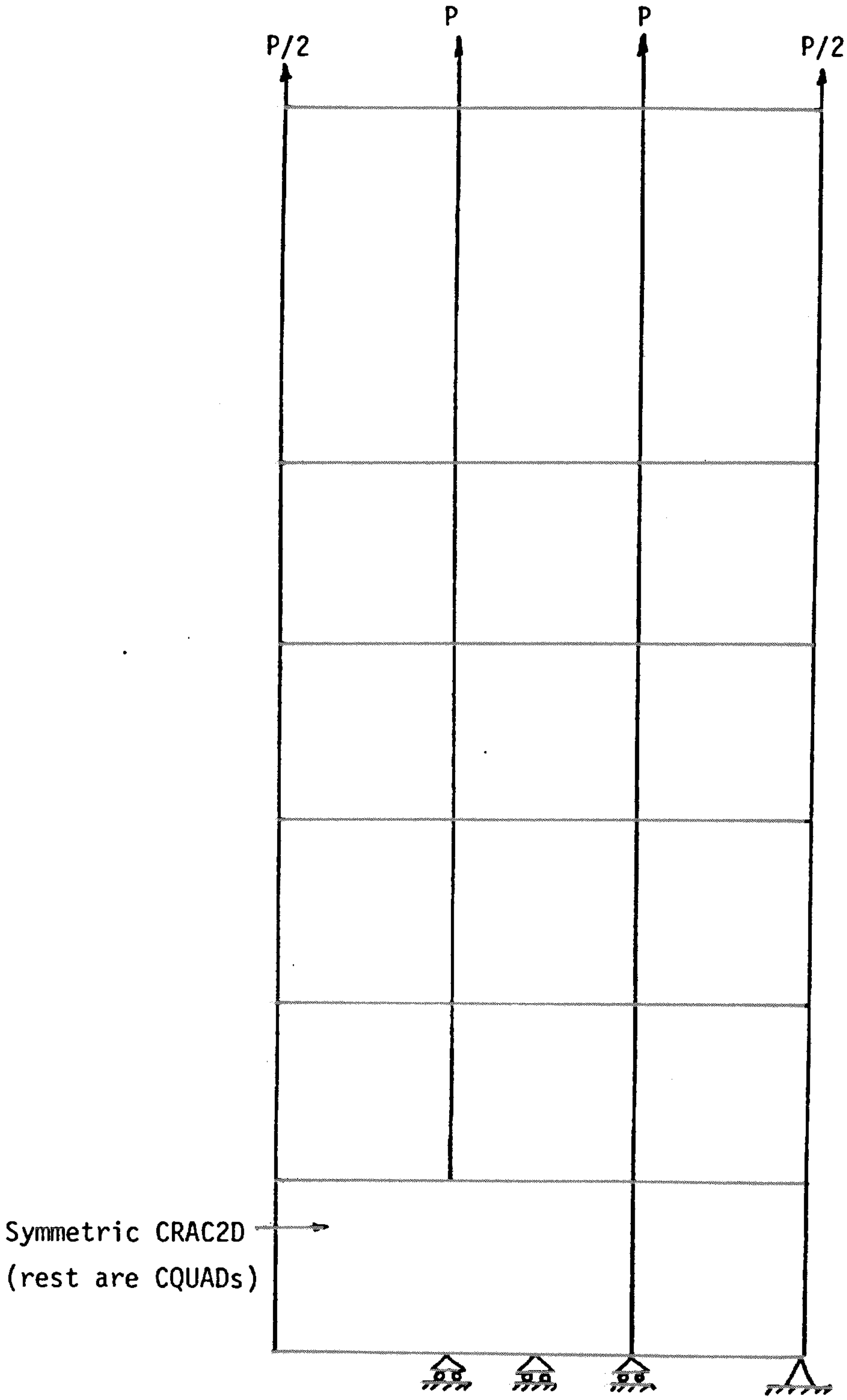
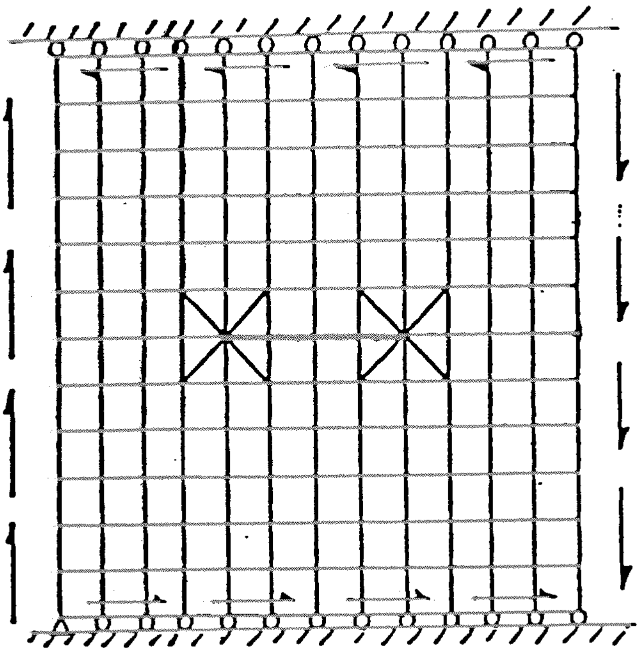
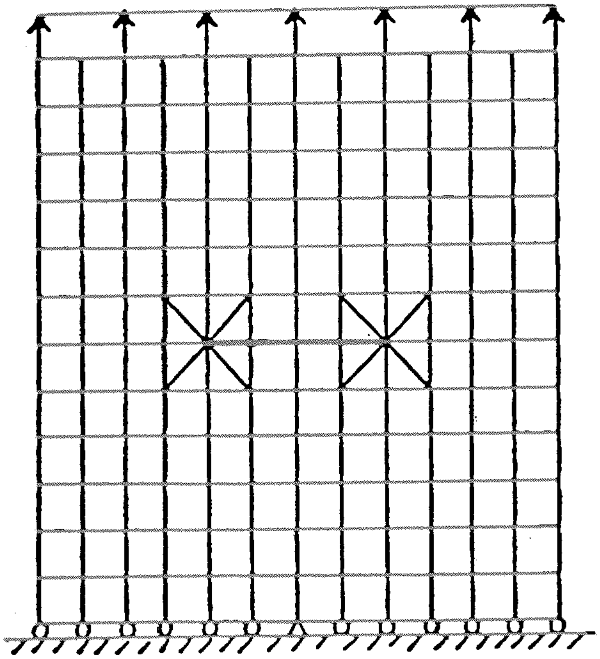


Figure 9. Model of Edge Crack with Uniform Load.



(a) loading condition for K_{II} calculation



(b) loading condition for K_I calculation

Figure 10. Model of Central Crack in Finite Plate.

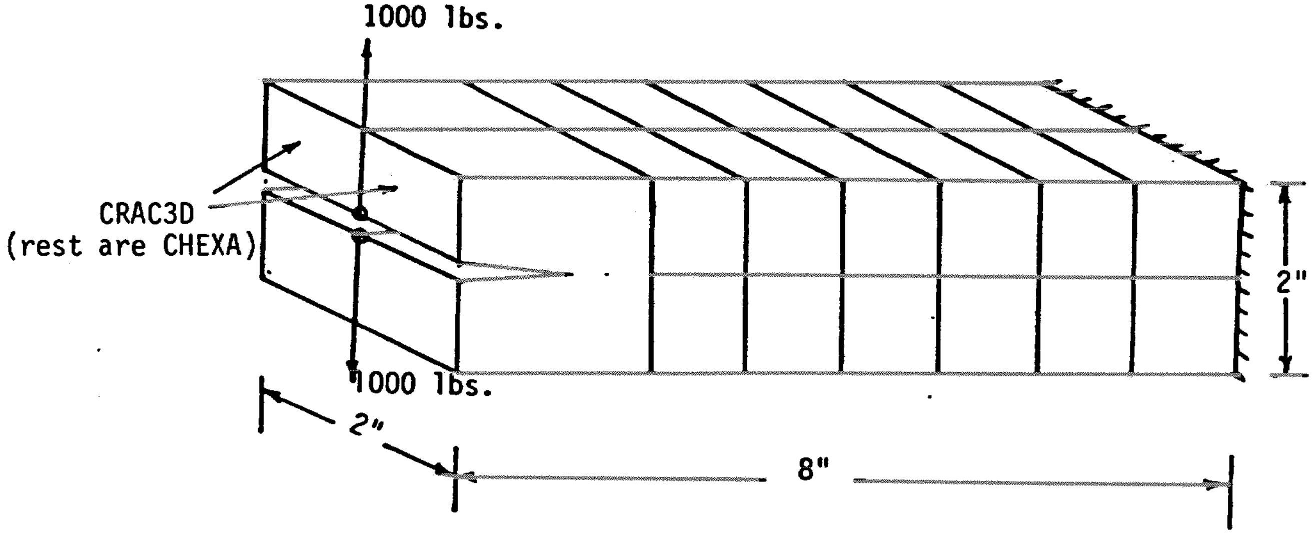


Figure 11. Double Cantilever Beam Problem.