SHAPE OPTIMAL DESIGN USING HIGH ORDER ELEMENTS

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Abstract

This paper presents some research results obtained recently in using the p-version of the Finite Element Method (FEM) for shape optimal design. The use of Bezier and B-spline curves to define design elements has proven to be an excellent way to model the geometry of the design problem. It is shown that the p-version 2-D elastic element can be extended to employ part of a Bezier or B-spline curve as its element side. This new element has been tested successfully with the patch test. Moreover, it is compatible, has no preferred direction and contains all the required rigid body modes (three zero eigenvalues are found in the element stiffness matrix).

Some classical shape optimal design problems have been tested using the CONLIN optimizer. Preliminary results indicate that similar optimal shapes can be obtained with fewer degrees of freedom than when compared to the h-version FEM. As with the h-version, ten iterations are sufficient for convergence in most of the problems. Extremely rapid convergence was observed when using lower order B-spline curves (4-5 order).

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1 INTRODUCTION

In any finite element analysis, discretization plays a critical role. Even for very simple problems unacceptable results will be obtained if the mesh is poorly designed. When shape optimal design is considered, finite element meshing becomes still a more important subject of concern. In a shape optimization process, the analyst usually starts with a rough geometrical description of a model and hopes to obtain an optimal shape with respect to the design variables he has selected. The final shape of the model, even though similar to the initial shape, will have completely different geometric aspect ratios. Therefore even the best mesh designed for the initial shape is usually not appropriate for the final shape, causing inacurracies in the computed stresses and displacements, as well as their sensitivity derivatives. As a result the "optimal" design generated by the optimization process might be completely meaningless.

Because the external boundary of the structure is modified at each stage of the optimization process, the finite element mesh must be appropriately updated in order to maintain the desired accuracy. In the conventional h-version of the Finite Element Method (FEM), it is therefore highly desirable, if not necessary, to include an adaptive mesh refinement scheme based on error estimates. Unfortunately a fully automated mesh adaptation capability is not yet at hand, and much research is still needed before incorporating such a tool within the shape optimization process. The task of maintaining the quality and the integrity of the finite element mesh becomes even more complicated when the analyst has to worry about allowable element distortion. In a general sense, an individual finite element will perform best when it is not distorted. For example the stresses and displacements computed from a quadrilateral element in a properly designed mesh will be quite accurate if the shape of the element is close to a rectangle of aspect ratio less than 3. The further the element is distorted into a trapezoid or a parallelogram, the further the deterioration of the computed results, with its interior angle being a reasonable measure of the deterioration. It is not obvious, in a shape optimization process, to avoid individual element distortion. As a result, it is desirable to employ elements that do not exhibit this sensitivity to shape.

An alternative to mesh adaptation techniques is to resort to an analysis model that rely more on a boundary representation. Recent progress in the p-version of the finite element method makes it possible to consider higher order elements for implementation into commercially available FEM systems such as MSC/NASTRAN. In these developments, high order elements were successfully tested in conjunction with linear blending mapping techniques [1,2]. It was demonstrated that very good results can be obtained in smooth problems when the model is discretized with only the minimum number of quadrangles. It has also been shown that with a high enough p-order the elements become less sensitive to shape, and displacements and stresses can be computed accurately in the entire domain. This nice matching between this methodology and the requirements of shape optimization was the motivation for further research.

The research results reported in this paper are concerned with shape optimal design of two-dimensional structures discretized with higher order elements. When used for shape optimal design applications, these elements offer many advantages in comparison with conventional low order elements. For example they facilitate considerably the definition of a suitable design model, because they mainly necessitate a boundary representation. The development effort started from a prototype FEM program that implements a 8-order quadrilateral element in conjunction with linear blending mapping techniques. The adaptation of this program to shape optimization has first required the consideration of other blending functions compatible with our geometric modeling scheme (Bezier, B-splines). Those blending functions are indeed needed in our approach to shape sensitivity analysis, which evaluates the objective function and constraint gradients through a finite difference technique. This approach to sensitivity analysis has proven to be quite efficient in a previous research project related to shape optimization using conventional low order elements [3].

After briefly reviewing how to properly describe the structural geometry using a small number of design variables (positions of control points), some basic properties of Bezier and B-spline curves will be explained and illustrated on simple examples. It will then be shown that the p-version 2-D elastic element can be extended to employ part of a Bezier or B-spline curve as its element side, provided that two simple rules are followed:

- the element order must be at least equal to the highest degree of its side curves;
- each element side should be defined by only one parametric curve.

This new element has been tested successfully with the patch test. Moreover, it is compatible, has no preferred direction and contains all the required rigid body modes (three zero eigenvalues are found in the element stiffness matrix). Some classical shape optimal design problems have been tested using the CONLIN optimizer. The results obtained up to now indicate that similar optimal shapes can be obtained with fewer degrees of freedom than when compared to the h-version FEM. As with the h-version, ten iterations are sufficient for convergence in most of the problems. Extremely rapid convergence was observed when using lower order B-spline curves (4-5 order).

2 GEOMETRIC DESIGN MODEL

The approach followed to describe the structural geometry is described in detail in previous papers [3,4,5], and can be summarized as follows. An internal parametric representation, typical of modern techniques employed in computer aided geometric design, is adopted. The structure is decomposed into a few subregions of simple geometry. These subregions are described in a compact way by using a limited number of control nodes. During the optimization process the geometry of conveniently selected subregions is allowed to change: these regions are called design elements. The movements of the corresponding control nodes are the design variables.

In this representation the FEM mesh can be directly derived from the coordinates of the control nodes. This feature leads to the distinction between a design model and an analysis model. The design model is made up of the small number of design elements, whose geometry is determined by the control node positions, and of the fixed subregions. By entering a relatively small number of design elements, it is possible to create a compact design model that describes well the structure to be optimized. The analysis model is the finite element model, characterized by the node coordinates of the mesh, the types and material properties of the elements, the applied loads and boundary conditions, etc... The analysis model can directly be derived from the design model at any stage of the iterative optimization process, because of the adopted internal parametric representation. This feature considerably facilitates the task of implementing the sensitivity analysis.

Two different types of parametric curves were investigated to define the design element boundaries. Bezier and B-spline curves are commonly used in Computer Aided Design (CAD) systems in order to develop complex geometric models (see e.g. Ref. [6]). Both types of curve are defined through the concept of control points, and they are therefore well suited for the description of our design model.

2.1 Bezier Curve

A Bezier curve is defined by the "vertices" of a polygon which uniquely defines the curve shape. The mathematical basis of the Bezier curve is the Bernstein function which is a polynomial blending function. The basis function is given by

$$J_{n,i}(t) = C_{n,i}t^i(1-t)^{n-i}$$

where

$$C_{n,i} = \frac{n!}{(i!(n-i)!)}$$

where n is the degree of the polynomial and i the particular vertex in the ordered set (from 0 to n). The curve points are given by

$$\vec{P}(t) = \sum_{i=0}^{n} \vec{P}_i J_{n,i}(t), \quad 0 < t < 1$$

where $\vec{P}_i$ represent the position vectors of the various vertices. In the two-dimensional case (2-D), $\vec{P}_i = (x_i, y_i)$. For 3-D problems, $\vec{P}_i = (x_i, y_i, z_i)$.

Bezier curves exhibit several interesting properties that can be summarized as follows. A nth degree Bezier curve is specified by n+1 vertices. The locations and the slopes of the starting and ending points of a Bezier curve are the same as those of the defining polygon. A Bezier curve is one parametric curve. A change in one vertex is felt throughout the entire curve. As an example, consider a four vertices cubic Bezier curve (n=3).

$$\begin{aligned}
J_{3,0}(t) &= C_{3,3}t^0(1-t)^{3-0} = (1-t)^3 \\
J_{3,1}(t) &= C_{3,3}t^1(1-t)^{3-1} = 3t(1-t)^2 \\
J_{3,2}(t) &= C_{3,3}t^2(1-t)^{3-2} = 3t^2(1-t) \\
J_{3,3}(t) &= C_{3,3}t^3(1-t)^{3-3} = t^3
\end{aligned}$$

$$\vec{P}(t) = (1-t)^3\vec{P}_0 + 3t(1-t)^2\vec{P}_1 + 3t^2(1-t)\vec{P}_2 + t^3\vec{P}_3$$

or

$$\vec{P}(t) = \vec{P}_0 + 3(\vec{P}_1 - \vec{P}_0)t + 3(\vec{P}_2 - 2\vec{P}_1 + \vec{P}_0)t^2 + (\vec{P}_3 - 3\vec{P}_2 + 3\vec{P}_1 - \vec{P}_0)t^3$$

2.2 B-spline Curve

A B-spline curve is also associated with the "vertices" of a polygon but it has a more flexible basis. This basis, called B-spline basis, is generally nonglobal and allows the order of the resulting curve to be defined independently of the number of defining polygon vertices. A B-spline basis of order n (degree n-1) which has a non-zero value between $t = t_i$ to $t = t_{i+1}$ is denoted as $B_{i,n}(t)$, where ti are elements of the knot vector which will be discussed below. Any B-spline basis $B_{i,n}(t)$ can be found by the recursive relation:

$$B_{i,n}(t) = \begin{cases} 1, & \text{if } n = 1 \text{ and } t_i < t < t_{i+1} \\ 0, & \text{if } n = 1 \text{ and } t_i > t > t_{i+1} \\ (t - t_i)/(t_{i+n-1} - t_i)B_{i,n-1}(t) + & \\ (t_{i+n-t})/(t_{i+n} - t_{i+1})B_{i+1,n-1}(t), & \text{if } n > 1 \end{cases}$$

The B-spline curve is given by

$$\vec{P}(t) = \sum_{i=0}^{k} \vec{P}_i B_{i,n}(t)$$

where $\vec{P}_i$ are the position vectors of the $k+1$ polygon vertices.

A knot vector is a series of integers ti, such that $t_i < t_{i+1}$ for all t_i. The values of ti are considered to be parametric knots. They are used to indicate the range of the parameter t. It is required to specify knots of multiplicity n at both the beginning and the end of the knot set for a curve of order n. A duplicate intermediate knot value indicates that a multiple vertex occurs at this point. Here we will only consider evenly spaced knots with unit separation between noncoincident knots. When there are no duplicate vertices, the parameter t varies from 0 to k-n+2 for an nth order curve defined by a k+1 vertices polygon. This means that the B-spline curve is constructed by k-n+2 parametric curves of order n which give n-1 order continuity all over the curve. Therefore the highest order with which a B-spline can be defined is k+1. the number of vertices of the control polygon; this special case generates a Bezier curve. A second order B-spline curve gives back the defining polygon. Different order B-spline curves defined by the same polygon, have the same end slopes. As an example, let us consider a third order (n=3, degree 2) B-spline curve defined by four vertices (k+1=4).

$$t_{max} = k - n + 2 = 2$$

The knot vector is (0 0 0 1 2 2 2). For $0 < t < 1$,

t_i	$B_{i,1}$	$B_{i,2}$	$B_{i,3}$
0	0	0	$(i-t)^2$
0	0	$1-t$	$t(1-t)+t(2-t)/2$
0	1	t	$t^2/2$
1	0	0	0
2	0	0	
2	0		
2			

The parametric equation for 0¡t¡1 is

$$\vec{P}(t) = \vec{P}_0 B_{0,3} + \vec{P}_1 B_{1,3} + \vec{P}_2 B_{2,3}$$

or

$$\vec{P}(t) = \vec{P}_0 + 2(\vec{P}_1 - (\vec{P}_0)t + (0.5\vec{P}_2 - 1.5\vec{P}_1 + \vec{P}_0 t^2)$$

which is quadratic and independent of $\vec{P}_3$. For $1 < t < 2$,

t_i	$B_{i,1}$	$B_{i,2}$	$B_{i,3}$
0	0	0	0
0	0	0	$(2-t)^2/2$
0	0	$2-t$	$t(2-t)/2+(2-t)(t-1)$
1	1	$t-1$	$(t-1)^2$
2	0	0	
2	0		
2			

The parametric equation for $1 < t < 2$ is

$$\vec{P}(t) = \vec{P}_1 B_{1,3} + \vec{P}_2 B_{2,3} + (\vec{P}_3) B_{3,3}$$

or

$$\vec{P}(t) = (2\vec{P}_1 - 2\vec{P}_2 + \vec{P}_3) + (-2\vec{P}_1 + 4\vec{P}_2 - 2\vec{P}_3)t + (0.5\vec{P}_1 - 1.5\vec{P}_2 + \vec{P}_3)t^2$$

which is quadratic and independent of P0.

It is important to realize that such a B-spline curve is constructed by two quadratic parametric curves. It has first order continuity throughout the entire curve but it suffers from second order discontinuity at its middle point.

3 ANALYSIS MODEL: HIGH ORDER ELEMENTS

The main distinguishing factor between the h-version and the p-version of the finite element method is that the shape functions used for element behavior in the p-version are not restricted to be linear or quadratic. On the contrary they can be defined with an arbitrary order p, with usually $2 \leq p \leq 8$. Moreover, in the p-version displacement FEM, the generalized coordinate associated with the displacement shape function does not stand for the displacement of a particular point. It is important to emphasize that the p-element is a parametric element. The principal idea of a parametric element is to model both the arbitrary geometry and its behavior through a mapping or a coordinate transformation in which the behavior in a simple region is distorted into a complex shape.

3.1 Modified p-Element

In the context of shape optimal design, p-elements are particularly interesting because they can almost completely match the "design elements" needed to construct the geometric model [4]. For this reason the geometry of any side of the 2-D elastic element was extended to use part of a Bezier or B-spline curve. The mapping technique used here is linear blending. By this method, the mapping of the interior points is uniquely determined by the four curves which define the boundary of the real domain. Since the newly extended curves are parametric curves, the parameter of the curve equation is directly used as the natural coordinate of the simple region or used through a linear transformation. So far, the mapping of the geometry of an element with a higher order boundary curve has been defined. Now we will give attention to the mapping of the behavior. The displacement shape functions used in this study are based on the integral of the Legendre polynomials. The displacement shape functions of a side mode for a quadrilateral element are integrals of the Legendre Polynomials. It is important to note that these shape functions are also parametric polynomial functions where the parameters are directly used as the natural coordinates of the simple region.

It will now be demonstrated why it is fundamental to have both the geometry and the behavior expressed as parametric polynomials and what might happen to the element properties when different orders are used for the geometry and for the behavior.

The original p-element from which the present research started, uses first or second order curve as element sides [1]. This element has all the basic element properties such as constant strain mode, three rigid body modes, and no preferred direction. Two fundamental rules have been found in using a higher order curve as an element side to keep all these properties:

- the element order should not be less than the highest degree of its side curves;
- any element side should be defined by only one parametric curve.

In the sequel some simple examples will be offered to illustrate why these rules have to be followed in order to keep the required properties. Figure 1 gives the structural layout which will be used in the following examples. The curve side between the two elements may be a Bezier curve or a 2nd degree B-spline curve depending on the example. Both curves have the same defining polygon which has 4 vertices, P0=(1.,0.), P1=(1.4,0.3), P2=(0.6,0.7), and P3=(1.,1.).

3.2 Constant strain mode

First, let us consider that the structure has a Bezier curve as inter-element side. Let the structure experience a constant strain test where the line AB moves uniformly in the x-direction by an amount d (assuming the structure has a zero Poisson's ratio i.e. no displacement in y-direction.) The deformed curve will remain a Bezier curve but the four vertices become $P_0 = (1. + 0.5d, 0.)$, $P_1 = (1.4 + 0.7d, 0.3)$, $P_2 = (0.6 + 0.3d, 0.7)$, and $P_3 = (1. + 0.5d, 1.)$. The displacement along the curve (x-direction) can be expressed as

$$u(t) = 0.5d + 0.6dt - 1.6dt^2 + 1.3dt^3$$

which is a parametric polynomial equation of degree 3. It is clear that a p-element of order 2 cannot represent this displacement field but a p-element of order 3 can represent it exactly. Therefore, for a p-element to have the constant strain mode, the order of the element should be no less than the highest degree of its side curves. This constitutes the first rule. Figure 2 shows the strain contour for this example with an order 2 element. When tested with an order 3 element, a constant strain mode was evident.

Now let us use a B-spline curve of degree 2 (order 3) defined by the same polygon to substitute for the Bezier curve. Again let the structure experience a constant strain test. If the structure do have the constant strain mode, then the deformed curve will remain a B-spline curve and the required displacement along the curve (x-direction) should be found as:

$$u(t) = 0.5d + 0.4dt - 0.4dt^2 \quad \text{for } 0 < t < 1$$
$$u(t) = 1.3d - 1.2dt + 0.4dt^2 \quad \text{for } 1 < t < 2$$

which are two quadratic parametric equations. This displacement field has first order continuity throughout the curve but it has a second order discontinuity at t=1. Since the displacement shape functions defined in this study do not have a second order discontinuity at their middle point, this displacement field will never be exactly matched. Thus the second rule is that an element side should be defined by only one parametric curve. Otherwise it will not have the constant strain mode. Tests were made on this example with the p-element orders ranging from 2 through 8. Their highest and lowest strains are given in Table 1.

3.3 Rotational rigid body mode

Exactly the same requirements have also been found for the element to present the rotational rigid body mode. For an element to have the rotational rigid body mode, its displacement shape functions should be able to represent its rotational displacement field exactly (infinitesimal rotation).

Considering again the example with a Bezier curve side, if the element is the mapping of the behavior. The displacement shape functions used in this study are based on the integral of the Legendre polynomials. The displacement shape functions of a side mode for a quadrilateral element are integrals of the Legendre Polynomials. It is important to note that these shape functions are also parametric polynomial functions where the parameters are directly used as the natural coordinates of the simple region.

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Now let us use a B-spline curve of degree 2 (order 3) defined by the same polygon to substitute for the Bezier curve. Again let the structure experience a constant strain test. If the structure do have the constant strain mode, then the deformed curve will remain a B-spline curve and the required displacement along the curve (x-direction) should be found as:

$$\begin{aligned} u(t) &= 0.5d + 0.4dt - 0.4dt^2 \quad \text{for } 0 < t < 1 \\ u(t) &= 1.3d - 1.2dt + 0.4dt^2 \quad \text{for } 1 < t < 2 \end{aligned}$$

which are two quadratic parametric equations. This displacement field has first order continuity throughout the curve but it has a second order discontinuity at t=1. Since the displacement shape functions defined in this study do not have a second order discontinuity at their middle point, this displacement field will never be exactly matched. Thus the second rule is that an element side should be defined by only one parametric curve. Otherwise it will not have the constant strain mode. Tests were made on this example with the p-element orders ranging from 2 through 8. Their highest and lowest strains are given in Table 1.

3.3 Rotational rigid body mode

Exactly the same requirements have also been found for the element to present the rotational rigid body mode. For an element to have the rotational rigid body mode, its displacement shape functions should be able to represent its rotational displacement field exactly (infinitesimal rotation).

Considering again the example with a Bezier curve side, if the element is rotated about the point P0 by an very small angle, the rotated curve side still is a Bezier curve. The displacement field along the curve side will again be a third order polynomial function. The difference is that displacements exist in both x and y directions. An element of order 3 or higher is required to represent the rotational rigid body mode in this example.

Now the side curve is considered to be a B-spline. Again the displacement field along the curve side now due to the rigid body rotation will have a second order discontinuity at the curve mid-point. Therefore this element will not have the rotational rigid body mode, even if order 8 is assumed. This supports the second rule presented above.

The eigenvalues of the element stiffness matrix are computed to check the existence of three rigid body modes. For all cases, there are always at least two zero eigenvalues which correspond to two translational rigid body modes. A comparison of the third eigenvalue is given in Figure 3 for the two cases mentioned above, as well as the case of an element with a circular side. A zero or nonzero third eigenvalue indeed shows up for all cases, as it should be expected. It is interesting to notice that the third eigenvalue of the circle side element converges to zero as the order of the element increases. The element with the newly extended curve side will have a zero third eigenvalue only if it follows the rules stated above.

4 SENSITIVITY ANALYSIS

The derivatives needed for solving the shape optimal design problem are computed by a finite difference scheme embedded into the FEM equation solver. In the p-version displacement FEM, the set of equilibrium equations used for the structural analysis can be written:

$$[K]\{C_u\} = \{C_f\}$$

where C_u denote the generalized coordinates associated with the displacement shape functions. C_f is the consistent load vector corresponding to the shape functions. $[K]$ is the global stiffness matrix.

Differentiating this set of equations with respect to the design variable xi, gives

$$[K]\{\frac{dC_u}{dx_i}\} = \{\frac{dC_f}{dx_i}\} - [\frac{dK}{dx_i}]\{C_u\}$$

This shows that the derivatives of the generalized coordinates, $\{dC_u/dx_i\}$, can be obtained by solving the original set of equations with another load vector $\{g_i\}$,the so-called pseudo-loads:

$$\{g_i\} = \{\frac{dC_f}{dx_i}\} - [\frac{dK}{dx_i}]\{C_i\}$$

Although the finite difference approach used in the sequel could be applied to get the derivatives of the consistent load vector $\{C_f\}$, for sake of simplicity, we shall assume that $\{C_f\} = 0$ in this paper.

The first derivatives of the stiffness matrix with respect to the design variables are calculated from the finite difference equations:

$$[\frac{dK}{dx_i}] = ([K(x_i + dx_i)] - [K(x_i)])/dx_i$$

where dx_i represents a small amount of perturbation. $[K(x_i + dx_i)]$ is the reassembled stiffness matrix of the structure with a perturbation dxi in the ith design variable x_i. Thus the pseudo-load can be formulated as

$$\begin{aligned}\{g_i\} &= ([K(x_i + dx_i)]\{C_u\} - [K(x_i)]\{C_u\})/dx_i \\ &= ([K(x_i + dx_i)]\{C_u\} - \{C_f\})/dx_i\end{aligned}$$

The derivatives of the generalized coordinates can now be obtained by solving the equilibrium equations with these additional pseudo-loads:

$$\{\frac{dC_u}{dx_i}\} = [K]^{-1}\{g_i\}$$

The number of pseudo-loads required is equal to the number of design variables times the number of loading conditions.

The solution $\{C_u(x_i + dx_i)\}$ which corresponds to the perturbated structure can be approximated by

$$\{C_u(x_i + dx_i)\} = \{C_u(x_i)\} + \{\frac{dC_u(x_i)}{dx_i}\}dx_i$$

This solution $\{C_u(x_i + dx_i)\}$ is used along with the shape functions which correspond to the perturbated structure to obtain the displacements $u(x_i + dx_i)$ and stresses $\sigma(x_i + dx_i)$ of the perturbated structure.

Then the derivatives of each displacement and stress functions are evaluated through the following finite difference scheme:

$$\frac{du}{dx_i} = (u(x_i + dx_i) - u(x_i))/dx_i$$

$$\frac{d\sigma}{dx_i} = (\sigma(x_i + dx_i) - \sigma(x_i))/dx_i$$

5 EXAMPLES OF APPLICATION

In this section, two typical design problems are offered to demonstrate the effectiveness of using the p-version FEM for shape optimization. Because of its many attractive features, the CONLIN optimizer was employed to solve the numerical optimization problem generated by the sensitivity analysis module. The convex linearization technique on which CONLIN relies has proven to be specially well suited for a broad class of structural optimization problems. This general and efficient optimizer benefits from some major advantages, which make it particularly well adapted to the present study (see Refs. [4,5])

5.1 Hole in a Biaxial Stress Field

A now classical test problem in shape optimal design is to find the best possible shape of a hole in an infinite plate under a biaxial stress field, such that stress concentration is minimized along the hole boundary. The problem considered is a square plate with a hole at its center. Uniform distributed forces are applied normal to the plate boundary. The force in the x-direction is twice that in the y-direction. We want to find a shape of the hole to minimize the maximum Von Mises stress of this structure. The theoretical solution of the original problem is an ellipse. Its axis ratio depends on the ratio of the two axial stresses but its size does not influence the stress concentration. The solution of the problem we posed is expected to be close to an ellipse but its size will tend to diminish because we are dealing with a finite plate. Therefore a constraint must be applied on the size of the hole.

Due to the symmetry of both the geometry and the load, only a quarter of the plate is modeled. The hole is described by a 7 node Bezier curve. The node locations along preassigned directions are considered as the design variables.

Figure 4 shows the layout of the structure and the nodal directions of freedom. To keep the tangency requirements at both ends of the curve, the end node and the node next to it are linked together as one design variable. Thus this problem has five design variables. The structure is modeled by 4 elements. The two elements along the hole form the design element. The other two belong to the fixed element. The problem is set up to minimize the maximum Von Mises stress in the structure subject to a bound on its weight (in order to limit the reduction of the hole). Since the maximum stress for this structure will always lie on the hole boundary (thickness is assumed uniform), eighteen evenly spaced points are picked along the curve boundary for monitoring the maximum Von Mises stress.

This problem is solved first with elements of order 8, the highest available order in the current implementation. The nice smooth stress distribution over the entire structure can be seen from the plot of the stress contours. Figures 5 and 6 show the stress contours of the initial and optimized structures, respectively. Note that one can hardly see the stress jump between the elements. Figure 7 gives the design history of the curve shape and its defining polygon, and it demonstrates the excellent convergence of the optimization process. The curves after 3 iterations are completely overlapped with each other. It is interesting to note that this convergence is even faster than when compared to the problem solved with the h-version FEM [3]. The reasons of this remarkable convergence speed may be because the stress is smooth over a large area and the stress constraints are set along the curve boundary.

The same problem was also solved with elements of order 2 through 7. Comparison of their optimized curve shapes and their defining polygons is given in Figure 8, which shows the nice convergence when increasing the element order. Since the final shape solved with element order 5 is already so close to that solved with element order 8, it is interesting to find out what is the maximum Von Mises stress when this structure is analyzed with element order 8. Thus we re-analyzed all the final structures with element order 8. The maximum Von Mises stresses found in optimizing with different orders are given in Table 2 along with those obtained by re-analyzing with element order 8. In Table 2, we also give the CPU time spent for one structural analysis and that for the full optimization process (all calculation were done on a IBM 4341). By comparing both the stress difference and the cost in CPU time, it seems that element order 6 is the best choice but the results of element order 5 are already very accurate. It should be noted that in this problem the element of order 5 does not have the rotational rigid body mode but since the curve is very smooth, this does not reduce its accuracy. The correct choice of element order will be further discussed in the next example.

5.2 Fillet

The fillet design problem is also a commonly used example in shape optimal design. In this problem a tension plate with a transition zone connecting two different width is considered. Due to the symmetry about both X and Y axes, only a quarter of the plate is considered. The actual structure being analyzed is given in Figure 9 along with its dimensions and notation. The segments S3 and S4 are the symmetric lines of the fillet. The boundary segment S1 is to be varied, with points at A and B fixed. A uniformly distributed load is applied on S2. The optimal design problem is again to find a boundary shape S1 to minimize the stress concentration, but no weight constraint is applied in this case. The boundary shape S1 is modeled by a 6 node Bezier curve. With the two end nodes fixed at A and B, the problem has 4 design variables. The structure is modeled with 6 p-elements. The two elements at the ends are fixed. The center part, which contains the boundary segment S1, is evenly divided into 4 p-elements to form the design element. The reason to introduce smaller elements in this part is that a more accurate stress prediction is desired along the curve boundary, where stress concentration is expected to occur. Again evenly spaced points along the curve boundary are picked for evaluating the maximum Von Mises stress.

This problem was also solved with all different p-element orders. the following results correspond to element order 8. The stress contours corresponding to the initial and final designs are shown in Figures 10 and 11. The initial high stress concentration at point B, Von Mises stress 294 N/mm^2, is smoothed out along the curve boundary with the maximum Von Mises stress being 146 N/mm^2 in the final design. The stress concentration factor is reduced from 2.45 to 1.22. The excellent convergence can be seen from its design history (Figure 12) where, after only four iterations, the design is completely converged.

Comparisons of the results of using different p-element orders are given in Figure 13 and Table 3. Figure 13 gives the final curve shapes and their defining that these rules do not serve as restrictions but rather as a guideline of how to properly define the higher order curve for an element side. A sensitivity analysis by finite difference was introduced for the p-version FEM, in order to calculate the derivatives of the structural behavior with respect to the shape changes. The CONLIN optimizer was adopted in our experimental program to solve some representative shape optimal design problems.

The advantages of using the p-version FEM for shape optimal design can be summarized as follows. The analysis model can be devised so that it exactly matches the design model. This avoids approximation errors between the design and analysis models. In the p-version FEM, the stresses along an element side are as accurate as those inside the element. This is important because generally the critical stress constraints are found along the design element boundary. Numerical experiments show that the error in the stresses, found using higher order p-elements, is insensitive to perturbations in the shape of the element. This means that the shape sensitivity derivatives calculated by a finite difference method are more accurate and reliable. These extremely accurate and stable sensitivity derivatives have been observed over a very wide range of perturbations (from 0.1

The effectiveness of using the p-version FEM in shape optimal design has been demonstrated in two typical examples. The accuracy of the optimization results was investigated by solving the problems with different element orders. Both the final optimized design and the minimized stress concentration converge smoothly to meaningful values when the element order is increased. These results prove the excellent accuracy of the proposed method. Finally, the variation of the cost in CPU-time with respect to the element order was investigated and an economical way to perform the optimization was suggested.

7 ACKNOWLEDGEMENT

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8 REFERENCES

1 K. Izadpanah, "Computation of Stress Components in the p-Version of the Finite Element Method", DSc. Dissertation, Washington University, St. Louis, 1985.

2 K. Izadpanah, "Evaluation of the p-Version of the Finite Element Method", MSC/NASTRAN User' Conference, March 1986.

3 C. Fleury and D. Liefooghe, "Shape Optimal Design on an Engineering Workstation", Paper presented at the MSC/NASTRAN Users Conference, Universal City, California, March 20-21, 1986.

4 C. Fleury, "Shape Optimal Design by the Convex Linearization Method", in The Optimum Shape: Automated Structural Design" (J. BENNETT and M. BOTKIN, eds.), 1986.

5 C. Fleury, "Computer Aided Optimal Design of Elastic Structures", in Computer Aided Optimal Design: Stuctural and Mechanical Systems (C.A. Mota Soares, ed), Springer Verlag, to appear, 1987.

6 W. Bohm, G. Farin, J. Kahmann, "A Survey of Curve and Surface Methods in CAGD", Computer Aided Geometric Design, Vol. 1, pp. 1-60, 1984.

Table 1. Strain/Stress for Various Element Orders

element order p	Strain in the x-dir. maximum ($\times 10^{-3}$)	Strain in the x-dir. minimum ($\times 10^{-3}$)	Von Mises stress maximum	Von Mises stress minimum
2	0.6095	0.3081	68.23	145.42
3	0.5174	0.4742	95.24	117.32
4	0.5255	0.4701	93.71	112.83
5	0.5085	0.4907	97.93	107.88
6	0.5106	0.4899	97.75	107.34
7	0.5049	0.4957	99.15	104.89
8	0.5054	0.4954	99.09	104.64

Table 2. Quarter Plate Problem - Summary of Results

element order p	maximum Von Mises Stress: results of order p	maximum Von Mises Stress: re-analysed with p=8	CPU-Time (sec.): one analysis	CPU-Time (sec.): full optimization
2	304.84	458.65	4.9	36.6
3	288.51	317.86	7.0	70.8
4	288.60	308.41	9.9	119.2
5	285.57	289.14	19.4	270.9
6	285.44	286.36	31.6	475.8
7	285.45	285.78	64.4	1007.9
8	285.42	-	103.4	1724.2

Table 3. Fillet Problem - Summary of Results

element order p	maximum Von Mises Stress		CPU-Time (sec.)	
	results of order p	re-analysed with p=8	one analysis	full optimization
2	134.94	197.43	10.0	62.7
3	138.80	182.07	13.5	123.1
4	141.77	169.35	17.9	210.4
5	142.78	159.01	31.8	480.4
6	144.82	148.92	50.2	855.2
7	145.77	147.58	110.3	1833.7
8	146.26	-	157.2	3081.4

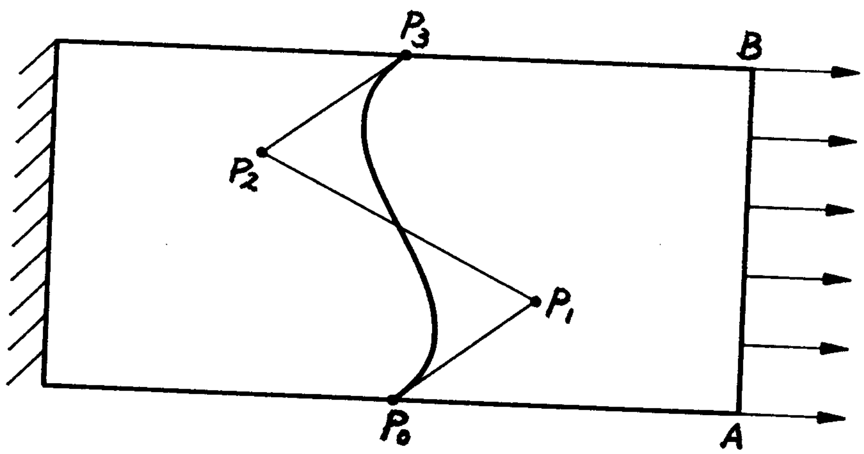


FIGURE 1. 2-ELEMENT STRUCTURAL LAYOUT

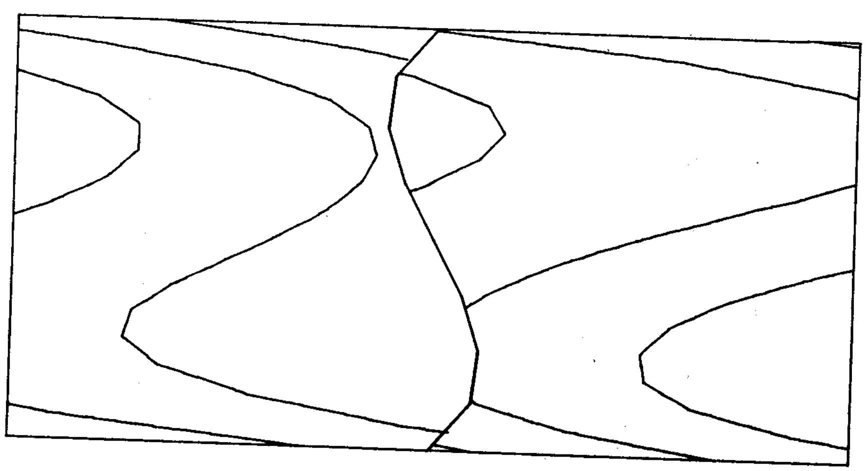
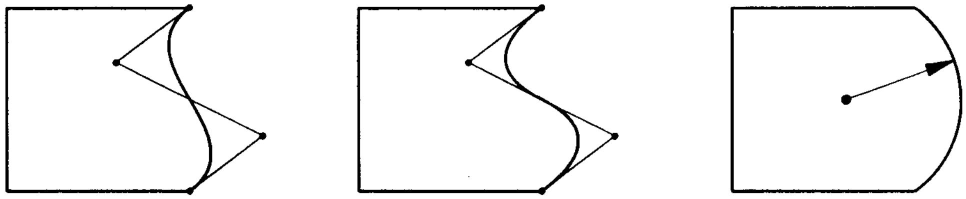


FIGURE 2. STRAIN CONTOURS (ORDER 2)



element order	4 Node Bezier (3rd degree)	4 Node B-spline (2nd degree)	Circle (infinite degree)
2	0.713E-01	0.102E+00	0.266E-02
3	-0.134E-14	0.744E-02	0.168E-03
4	-0.131E-14	0.566E-02	0.883E-06
5	-0.203E-14	0.132E-02	0.211E-07
6	-0.242E-14	0.127E-02	0.462E-10
7	-0.343E-14	0.380E-03	0.570E-12
8	-0.294E-14	0.362E-03	-0.135E-14

Figure 3. The Third Eigenvalue of Element Stiffness Matrix

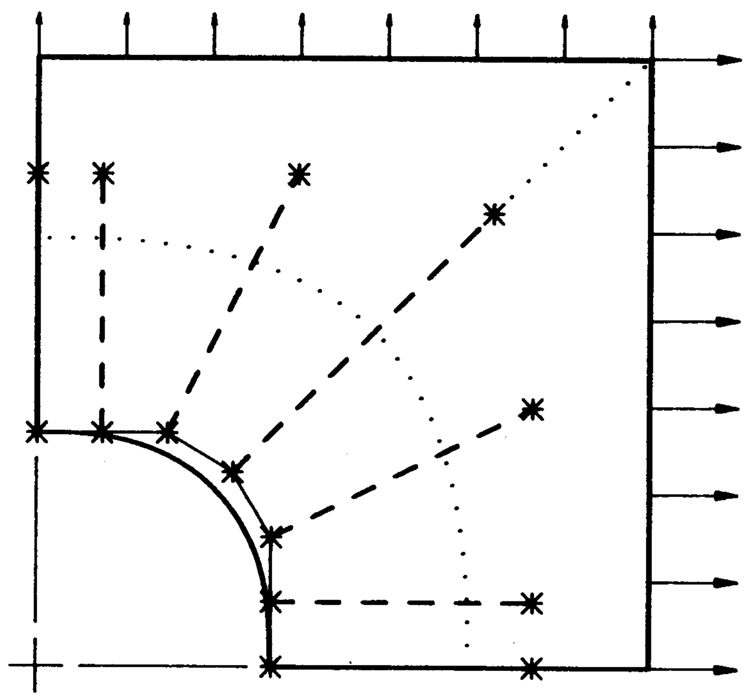


FIGURE 4. DESIGN MODEL FOR QUARTER PLATE PROBLEM

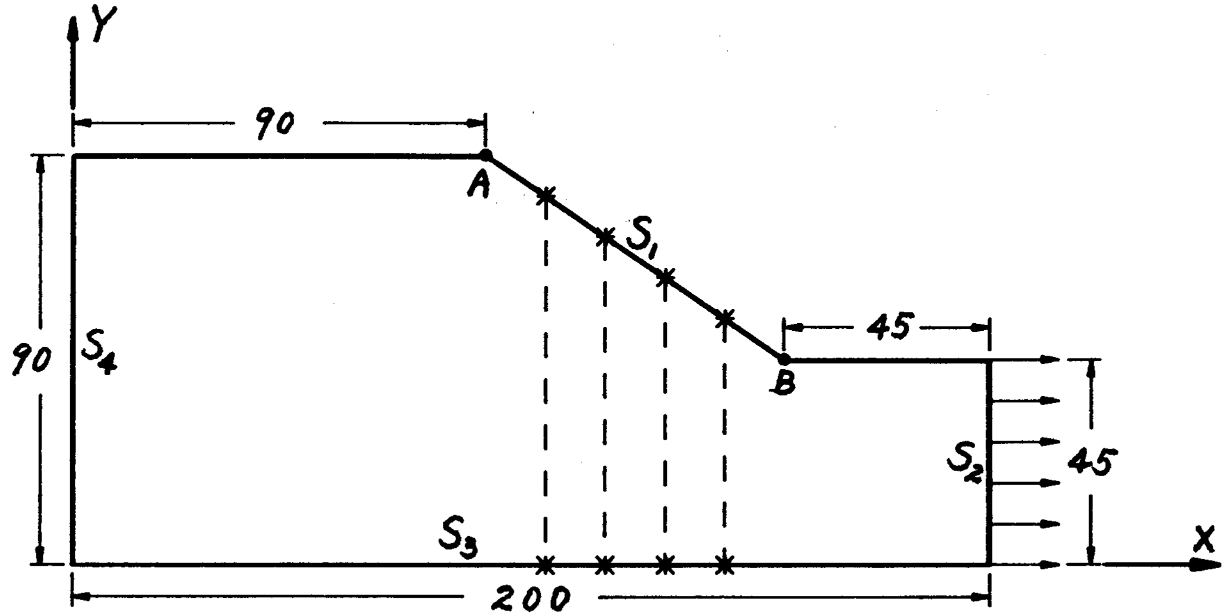


FIGURE 9. DESIGN MODEL FOR FILLET PROBLEM

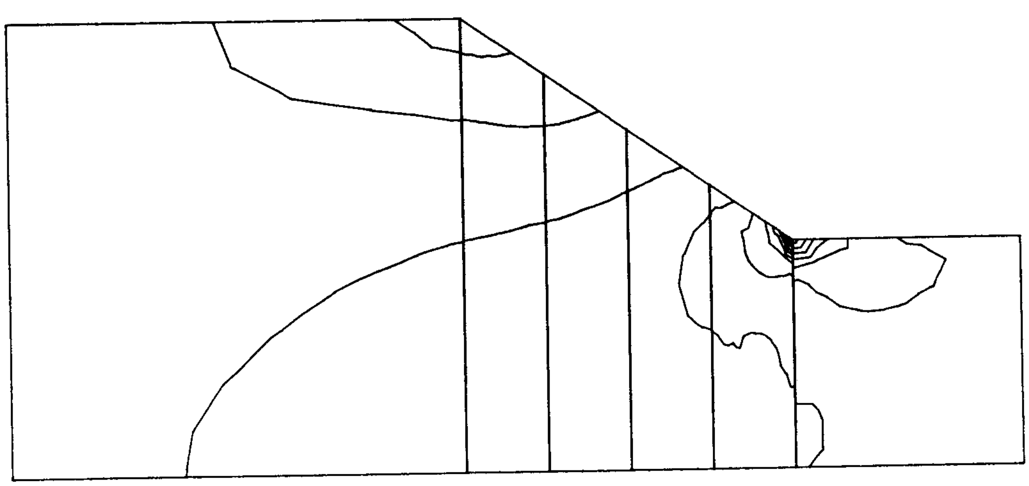


FIGURE 10. VON MISES STRESS - INITIAL DESIGN

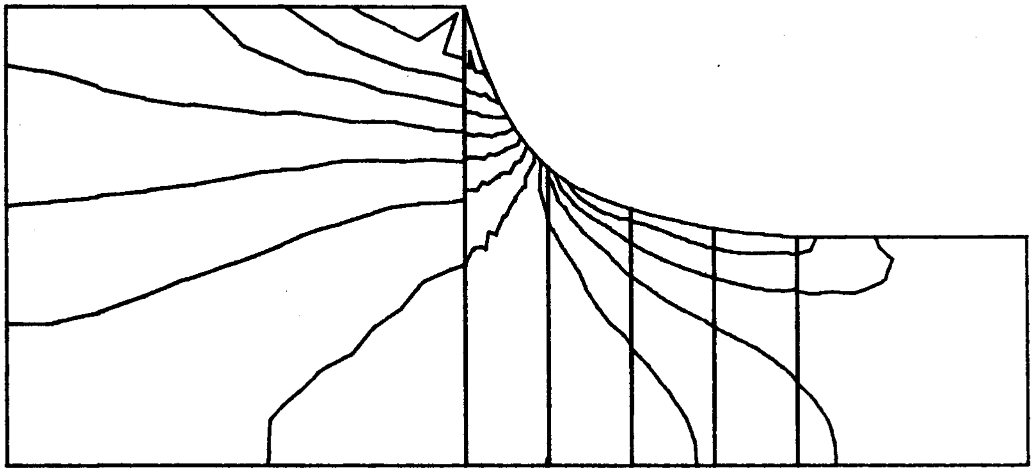


FIGURE 11. VON MISES STRESS - FINAL DESIGN

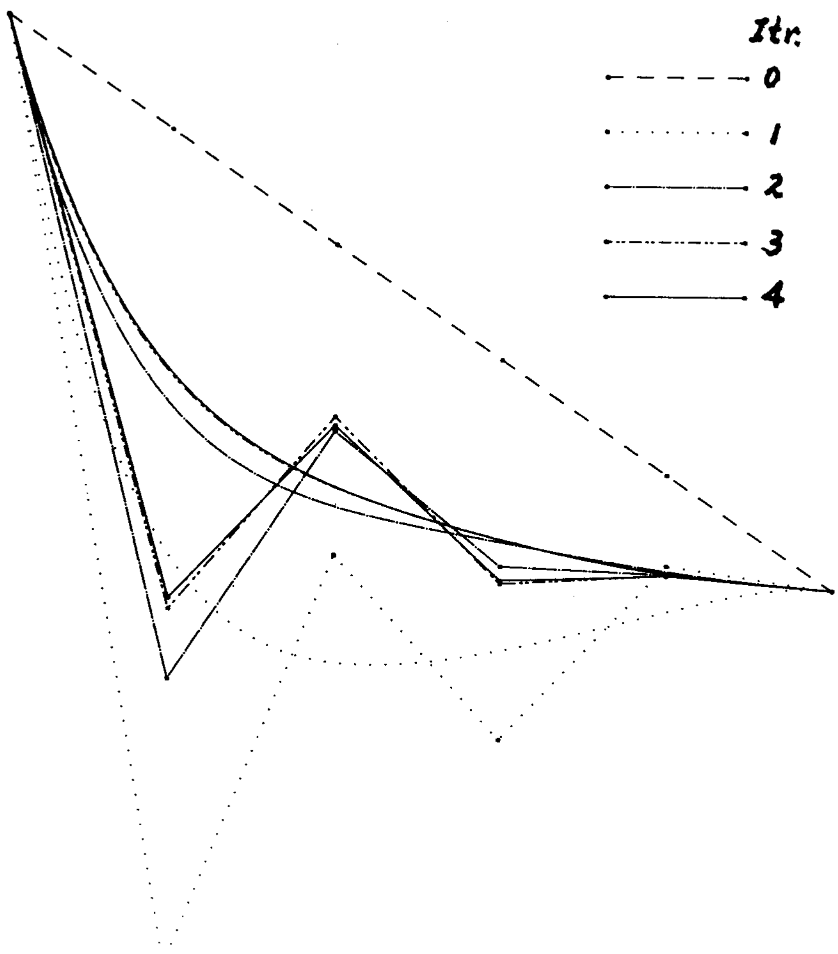


FIGURE 12. ITERATION HISTORY FOR FILLET PROBLEM

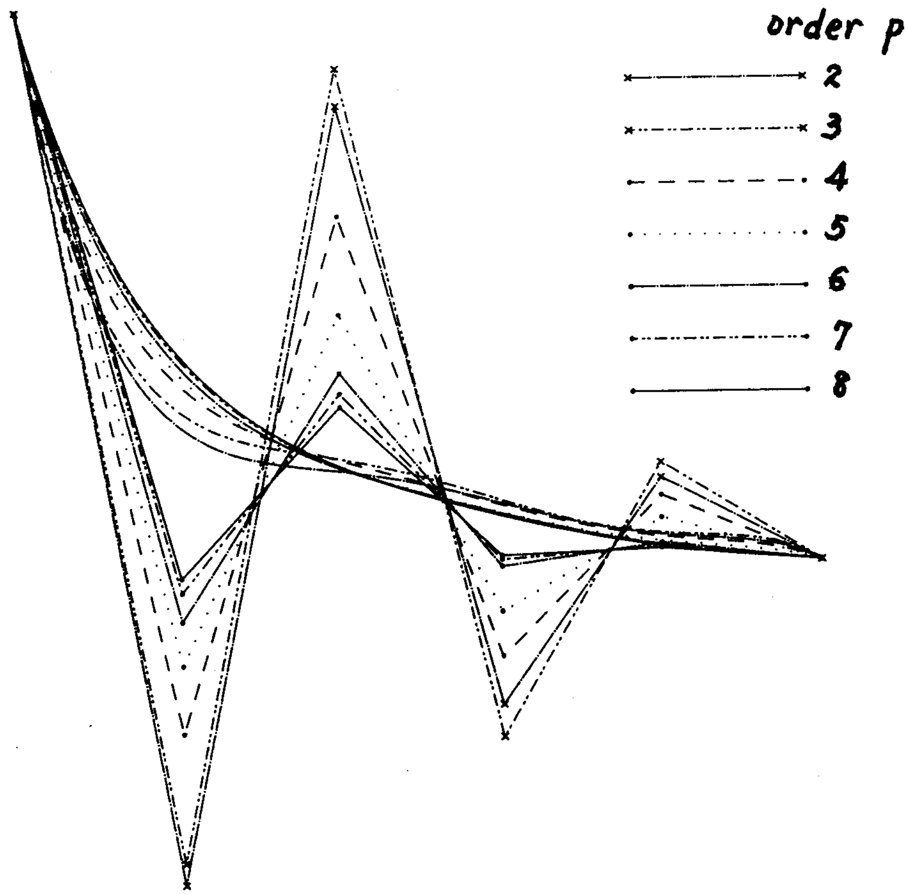


FIGURE 13. FINAL DESIGNS FOR VARIOUS p-ORDERS

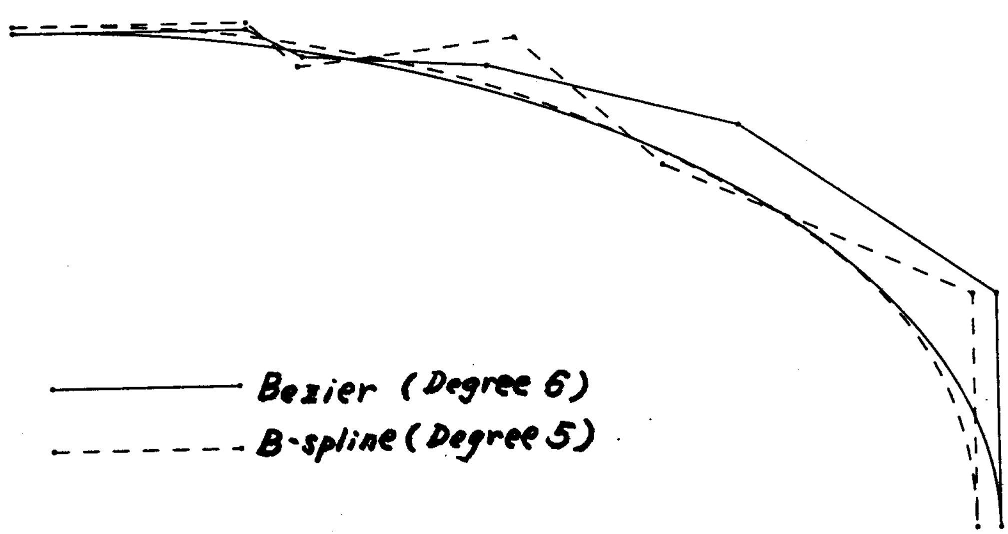


FIGURE 14. COMPARISON OF BEZIER AND B-SPLINE RESULTS