A Method for Predicting The Output Cross Power Spectral Density Between Selected Variables in Response to Arbitrary Random Excitations

by

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Reference: (1) DYNAMICS OF STRUCTURES, R. W. Clough and J. Penzien, McGraw-Hill, 1975.

SUMMARY:

A derivation of the response of lumped parameter, linear systems to random vibration excitations has been provided in order to extend the method to enable the prediction of the output cross power spectral density existing between various degrees of freedom.

DISCUSSION:

The response of linear, lumped parameter systems to general excitations can be obtained by either of two methods: 1) the direct integration of the equations of motion, or 2) the expansion of decoupled equations involving generalized coordinates (normal modes of vibration). The solution by the latter method requires that damping terms be expressible in terms of either a multiple of the stiffness or mass matrices, or a combination thereof. It is usual practice to solve the equations assuming that damping is small, exists separately for each mode, and that the modes and frequencies can be obtained within desired accuracy ignoring the effect of such damping. In view of the sparsity of information concerning the damping phenomenon and the usual magnitude of modal damping, these assumptions are adequate for the analysis of the response of metallic type structures.

The finite element method of discretizing continuous structures results in a lumped parameter system which can be analyzed by the above methods. The MSC/NASTRAN computer program provide a means of obtaining both direct as well as modal solutions for complex structures subjected to arbitrary excitations. In particular, the rigid format frequency response solution 30 provides for the post-processing of frequency response data to obtain output responses of various specified variables for response to Gaussian random vibration excitation. Also, the inputs to various points can be correlated arbitrarily as the user desires. However, the cross power spectral density existing between output quantities is not provided.

It is the purpose of this paper to present a method whereby the desired cross power spectral density between output displacements can be determined. Appendix A provides the derivation. The results show that it is possible to obtain the cross correlation existing between two output variables by creating two new scalar variables, equal to the sum and difference of the desired output variables. Then, the output cross power spectral density is given by one fourth of the difference in the power spectral density of the scalar responses provided by the "OUTPUT(XYOUT)" case control request in the SOLUTION 30 rigid format solution, i.e.:

Let: $v_1(t)$ = first output variable

$v_2(t)$ = second " "

Define new scalar variables: $S_3(t) = v_1(t) + v_2(t)$

$S_4(t) = v_1(t) - v_2(t)$

Let Phi_3 = PSD of response of S_3

Phi_4 = " " " " S_4

Then:

$$Phi_{12} = \frac{Phi_3 - Phi_4}{4}$$

= the cross power spectral density between v1 and v2

Figures 1 through 3 illustrate the application to a specific simple example problem. The problem consists of a cantilever beam, to which several bars are attached. The variables of interest are the differences in the displacements of points 6 and 7, which are located a small distance apart. The configuration is offset 45 degrees in the xy plane from the coordinate axes to form a skewed system. Figure 1 shows the geometry, and the results of the modal analysis. Thus, we define new scalar variables as follows, using multiple point constraint (MPC) equations:

Scalar variable 20 = del. x = $x_6 - x_7$

" " 21 = del. y = $y_6 - y_7$

The ordinary solution 30 output will give the power spectral density of the output displacements for the new scalar variables. However, the cross PSD must be determined by defining other new scalar variables, as follows:

Scalar variable 22 = del. x + del. y

" " 23 = del. x - del. y

The output of solution 30 will also yield the PSD's of 22 and 23, as well as the rms values. Figures 2 and 3 show the corresponding PSD diagrams. The cross PSD's can be determined simply as one fourth of the difference of the figures, and the rms value as one half of the square root of the difference in the mean square values.

APPENDIX A

Random Response of Linear Systems

Random Response of Linear Systems

The equations of motion of a linear, lumped parameter system may be written in matrix notation, as follows:

Let $[\ \ ]$ denote a square matrix

$\{\ \ \}$ denote a vector

$[M]$ denote a mass matrix

$[K]$ denote a stiffness matrix

$[C]$ denote a damping matrix

$\{P(t)\}$ denote a force vector

$\{\dot{X}\}$ denote a differentiation with respect to time

Thus

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{P(t)\} \qquad (1)$$

This set of n linear differential equations can be decoupled when the damping terms are considered to be proportional to stiffness, or mass, or a combination of the two ('Rayleigh' damping). Alternatively, a solution can be obtained for the undamped case, and damping applied separately to each decoupled coordinate. Hence, if this latter approach is followed, we obtain the reduced set of equations.

$$[M]\{\ddot{x}\} + [K]\{x\} = \{P(t)\} \qquad (2)$$

The solution of the homogeneous set of equations

$$[M]\{\ddot{X}\} + [K]\{X\} = 0 \qquad (2a)$$

leads to the eigensolution

$$|K - \lambda M| = 0 \qquad (3)$$

Denoting eigenvalues by ω_n^2 and eigenvectors by ϕ_n, the solution can be obtained in terms of two matrices; i.e,

$$\text{The spectral matrix} = [\Omega]^2 = \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots \ \omega_n^2 \end{bmatrix} \qquad (4)$$

$$\text{And the modal matrix} = [\Phi] = [\{\phi_1\}\{\phi_2\}\cdots\{\phi_n\}] \qquad (5)$$

The eigenvalues can be expressed in terms of Generalized Quantities, as $\omega_j^2 = k_j/M_j$

where: k_j = generalized stiffness in j^{th} mode

$$= \{\phi_j\}^T[K]\{\phi_j\}$$

M_j = generalized mass in j^{th} mode

$$= \{\phi_j\}^T[M]\{\phi_j\} \equiv 1$$

(For orthonormalization with respect to mass)

Thus, equation (1) can be expressed in terms of the generalized (modal) coordinates as:

$$\ddot{X}_n(t) + 2\omega_n \zeta_n \dot{X}_n(t) + \omega_n^2 X_n(t) = \frac{P_n(t)}{M_n} ; \; n = 1, 2, \dots \quad (6)$$

where

X_n = generalized displacement in n^{th} mode

ζ_n = percent of critical damping in n^{th} mode

$P_n(t)$ = generalized force in n^{th} mode

$= [\phi]^T \{P_n(t)\}$

The solution for the original coordinates can be found from the expansion of the solution of the response of the generalized coordinates, as:

$$\{X\}_{n\times 1} = [\Phi]_{n\times n} \{X(t)\}_{n\times 1} \quad (7)$$

where

$$\{X(t)\} = \begin{Bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{Bmatrix}$$

Convergence of Eqn. (7) can usually be obtained by truncation of the higher order terms in (7), due to the fact that the contribution of the higher order modes diminishes with increased damping.

Now, consider any original coordinate $X_j(t)$. From (7)

$$X_j(t) = \langle \phi_j \rangle_{1\times n} \{X(t)\}_{n\times 1} = \sum_n \phi_{mj} X_m(t) \quad (8)$$

Assuming that the linear, lumped parameter system is exposed to a stationary, Gaussian random process, then the auto correlation function of the response is given by:

$$R_{X_j}(t) = E\left\{X_j(t)\,X_j(t+T)\right\} = E\left\{\sum_m\sum_n \phi_{mj}\,\phi_{nj}\,X_m(t)\,X_n(t+T)\right\} \quad (9)$$

But, Eqn. (6) can be solved for any modal response by the use of the Duhamel Integral, which leads to the response solution:

$$X_n(t) = \frac{1}{M_n\omega_n(1-\zeta^2)^{1/2}}\int_0^t P_n(T)\,e^{-\zeta_n\omega_n(t-T)}\sin\left[\omega_n(1-\zeta_n^2)^{1/2}(t-T)\,dT\right] \quad (10)$$

or

$$X_n(t) = \int_0^t P_n(T)\,h_n(t-T)\,dT \quad (11)$$

where,

$$h_n(t) = \frac{1}{\omega_{D_n} M_n}\,e^{-\zeta_n\omega_n t}\sin\omega_{D_n} t \quad (12)$$

where:

$$\omega_{D_n} = \omega_n(1-\zeta_n^2)^{1/2}$$

Substituting (11) into (9) gives:

$$R_{X_j}(T) = E\left[\sum_m\sum_n\int_0^t\int_0^{t+T}\phi_{mj}\,\phi_{nj}\,P_m(\theta_1)\,P_n(\theta_2)\,h_m(t-\theta_1)\,h_n(t-\theta_2+T)\,d\theta_1\,d\theta_2\right] \quad (13)$$

Letting $\upsilon_1 = t-\theta_1$ and $\upsilon_2 = t-\theta_2+T$

$$d\upsilon_1 = -d\theta_1 \qquad d\upsilon_2 = -d\theta_2$$

then $R_{X_j}(T) = \sum_m\sum_n R_{X_{jm}X_{jn}}(T)$ (14)

where

$$R_{X_{jm}X_{jn}}(T) = \int_0^\infty\int_0^\infty R_{P_mP_n}(t-\upsilon_2+\upsilon_1)\,h_m(\upsilon_1)\,h_n(\upsilon_2)\,d\upsilon_1\,d\upsilon_2 \quad (15)$$

where

$R_{P_mP_n}(T)$ = covariance function for loading variables $P_m(t)$ and $P_n(t+T)$

ANALYSIS

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and

$R_{X_{jm}X_{jn}}(\tau)$ = covariance function for modal responses $X_{jm}(t)$ and $X_{jn}(t)$

The power spectral density of $X_j(t)$ is the Fourier Transform of $R_{X_j}(t)$; i.e

$$S_{X_j}(\bar\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{X_j}(\tau)\, e^{-i\omega\tau}\, d\tau \qquad (16)$$

From (14) thru (16):

$$S_{X_j}(\bar\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_m\sum_n\int_0^{\infty}\int_0^{\infty}\phi_{mj}\,\phi_{nj}\,R_{P_mP_n}(t-v_2+v_1)\,h_m(v_1)\,h_2(v_2)\,dv_1\,dv_2\right]e^{-i\omega\tau}\,d\tau$$

$$= \frac{1}{2\pi}\sum_m\sum_n\left[\lim_{T\to\infty}\int_0^T h_m(v_1)\,e^{i\bar\omega v_1}\,dv_1\int_0^T h_n(v_2)\,e^{-i\bar\omega v_2}\,dv_2\right]S_{P_mP_n}(\bar\omega) \qquad (17)$$

where

$$S_{P_mP_n}(\bar\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{P_mP_n}(\tau)\, e^{-i\omega\tau}\, d\tau \qquad (18)$$

Now, superposition of stationary processes leads to the relation:

$$S_{X_j}(\bar\omega) = \sum_m\sum_n S_{X_{jm}X_{jn}}(\bar\omega) \qquad (19)$$

Hence, Eqn. (17) can be written as (19), with

$$S_{X_{jm}X_{jn}}(\bar\omega) = \phi_{mj}\,\phi_{nj}\,H_m(-i\bar\omega)\,H_n(i\bar\omega)\,S_{P_mP_n}(\bar\omega) \qquad (20)$$

where

$$H_n(i\bar\omega) = \int_{-\infty}^{\infty} h_n(t)\,e^{-i\bar\omega t}\,dt = \frac{1}{M_n\left[\omega_n^2 + 2i\zeta_n\bar\omega\omega_n - \omega^2\right]} \qquad (21)$$

for $0 < \zeta_n < 1$

It is to be noted that for $\zeta_n \ll 1$ the terms in Equation (19)

$$S_{\dot{x}_{jm}\dot{x}_{jn}}(\bar{\omega}) \cong 0 \quad , \text{ for } m \neq n .$$

Thus

$$S_{\dot{x}_j}(\bar{\omega}) \cong \sum_n S_{\dot{x}_{jn}\dot{x}_{jn}}(\bar{\omega}) \tag{23}$$

where

$$S_{\dot{x}_{jn}\dot{x}_{jn}}(\bar{\omega}) = \phi_{nj}^2 \left| H_n(i\bar{\omega}) \right|^2 S_{P_nP_n}(\bar{\omega})$$

$$= \left| H_{nj}(i\bar{\omega}) \right|^2 S_{P_nP_n}(\bar{\omega}) \tag{24}$$

Equation (23) is the familiar sum square relation for superposition of modal responses.

Now, the cross-PSD of the loading function can be derived as follows.

$$P_m(t) = \{\phi_m\}^T \{P(t)\}$$

$$P_n(t) = \{\phi_n\}^T \{P(t)\} \tag{25}$$

Thus,

$$P_m(t)\,P_n(t) = \{\phi_m\}\{P(t)\}\{P(t)\}^T\{\phi_n\} \tag{26}$$

Therefore,

$$S_{P_mP_n}(\bar{\omega}) = \{\phi_m\}^T S_p(\bar{\omega})\{\phi_n\} \tag{27}$$

where

$$S_p(\bar{\omega}) = S\left[\{P(t)\}\{P(t)\}^T\right] \tag{28}$$

For any mode, the motion of any point 'k' is related to the motion of point 'j' by the eigenvector for that mode; i.e.,

$$X_{k_m}(t) = \left(\frac{\phi_{mk}}{\phi_{mj}}\right) X_{j_m}(t)$$

Thus

$$\ddot{X}_{k_m}(t) = \left(\frac{\phi_{mk}}{\phi_{mj}}\right) \ddot{X}_{j_m}(t)$$

The power spectral density will then be

$$S_{k_{mm}}(\bar{\omega}) = \left(\frac{\phi_{mk}}{\phi_{mj}}\right)^2 S_{j_{mm}}(\bar{\omega}) \tag{29}$$

Also, the cross-PSD will be given by:

$$S_{j_m k_m}(\bar{\omega}) = \left(\frac{\phi_{mk}}{\phi_{mj}}\right) S_{j_{mm}}(\bar{\omega}) \tag{30}$$

Substituting from Eqn. (24)

$$S_{j_m k_m}(\bar{\omega}) = \left(\frac{\phi_{mk}}{\phi_{mj}}\right) \phi_{mj}^2 \left[H_m(i\bar{\omega}) \times H_m(-i\bar{\omega})\right] S_{P_m P_m}(\bar{\omega})$$

$$= \phi_{m_k} \phi_{mj} \left[H_m(i\bar{\omega}) \times H_m(-i\bar{\omega})\right] S_{P_m P_m}(\bar{\omega}) \tag{31}$$

Summing the modal responses gives:

$$S_{jk}(\bar{\omega}) = \sum_n S_{j_m k_m}(\bar{\omega}) \tag{32}$$

where

$$S_{j_m k_m}(\bar{\omega}) = \left[H_{m_j}(i\bar{\omega}) \times H_{m_k}(-i\bar{\omega})\right] S_{P_m P_m}(\bar{\omega}) \tag{33}$$

In the input PSD and transfer functions are superimposed to create composite functions, then Eqns. (23) and (32) simplify to:

$$S_j(\bar{\omega}) = \left|H_j(i\bar{\omega})\right|^2 S_i(\bar{\omega}) \tag{34}$$

$$S_{jk}(\bar{\omega}) = \left[H_j(i\bar{\omega}) \times H_k(-i\bar{\omega})\right] S_i(\bar{\omega}) \tag{35}$$

where

$S_i(\bar{\omega})$ = composite input PSD

Eqn (35) shows that it is possible to obtain the cross PSD of the responses of any two coordinates by permuting their transfer functions and multiplying by the input PSD. It is also possible to obtain the mean value of the response by a different technique. We create new scalar coordinates equal to the sum and difference of desired coordinates as follows.

let

$$\begin{aligned} S_k(t) &= v_1(t) + v_2(t) \\ S_\ell(t) &= v_1(t) - v_2(t) \end{aligned} \tag{36}$$

Then, the transfer functions of these scalar variables will be given by:

$$\begin{aligned} H_k(i\bar{\omega}) &= H_1(i\bar{\omega}) + H_2(i\bar{\omega}) \\ H_\ell(i\bar{\omega}) &= H_1(i\bar{\omega}) - H_2(i\bar{\omega}) \end{aligned} \tag{37}$$

The mean square value of the responses will be given by:

$$\begin{aligned} \sigma_{S_k}^2 &= \int_{-\infty}^{\infty} \left|H_k(i\bar{\omega})\right|^2 S_i(\bar{\omega})\, d\bar{\omega} \\ \sigma_{S_\ell}^2 &= \int_{-\infty}^{\infty} \left|H_\ell(i\bar{\omega})\right|^2 S_i(\bar{\omega})\, d\bar{\omega} \end{aligned} \tag{38}$$

Taking the difference of Eqn (38), we obtain:

$$\nabla_{hl} = \sigma_{S_h}^2 - \sigma_{S_l}^2 \tag{39}$$

let

$$H_1(i\bar{\omega}) = a_1(\bar{\omega}) + ib_1(\bar{\omega})$$
$$H_2(i\bar{\omega}) = a_2(\bar{\omega}) + ib_2(\bar{\omega}) \tag{40}$$

Substitute into (37):

$$H_h(i\bar{\omega}) = a_1(\bar{\omega}) + a_2(\bar{\omega}) + i\left[b_1(\bar{\omega}) + b_2(\bar{\omega})\right]$$
$$H_l(i\bar{\omega}) = a_1(\bar{\omega}) - a_2(\bar{\omega}) + i\left[b_1(\bar{\omega}) - b_2(\bar{\omega})\right] \tag{41}$$

Substitute into (38):

$$\sigma_{S_h}^2 = \int_{-\infty}^{\infty}\left\{\left[a_1(\bar{\omega}) + a_2(\bar{\omega})\right]^2 + \left[b_1(\bar{\omega}) + b_2(\bar{\omega})\right]^2\right\} S_i(\bar{\omega})\,d\bar{\omega}$$
$$\sigma_{S_l}^2 = \int_{-\infty}^{\infty}\left\{\left[a_1(\bar{\omega}) - a_2(\bar{\omega})\right]^2 + \left[b_1(\bar{\omega}) - b_2(\bar{\omega})\right]^2\right\} S_i(\bar{\omega})\,d\bar{\omega} \tag{42}$$

Substitute into (39):

$$\nabla_{hl} = 4\int_{-\infty}^{\infty}\left[a_1(\bar{\omega}) \times a_2(\bar{\omega}) + b_1(\bar{\omega}) \times b_2(\bar{\omega})\right] S_i(\bar{\omega})\,d\bar{\omega} \tag{43}$$

<u>But</u>, substitute (40) into (35) gives:

$$S_{12}(\bar{\omega}) = \left[a_1(\bar{\omega}) + ib_1(\bar{\omega})\right]\left[a_2(\bar{\omega}) - ib_2(\bar{\omega})\right] S_i(\bar{\omega})$$

and

$$\sigma_{12}^2 = \int_{-\infty}^{\infty} S_{12}(\bar{\omega})\,d\bar{\omega}$$
$$= \int_{-\infty}^{\infty}\left\{\left[a_1(\bar{\omega}) \times a_2(\bar{\omega}) + b_1(\bar{\omega}) \times b_2(\bar{\omega})\right] + i\left[b_1(\bar{\omega})a_2(\bar{\omega}) - b_2(\bar{\omega})a_1(\bar{\omega})\right]\right\} S_i(\bar{\omega})\,d\bar{\omega} \tag{44}$$

Now, the integral in (44) can be written as:

$$\sigma_{12}^2 = \int_{-\infty}^{\infty} S(\bar\omega)\, d\bar\omega$$

$$= \int_0^{\infty} S(\bar\omega)\, d\bar\omega + \int_{-\infty}^{0} S(\bar\omega)\, d\bar\omega$$

$$= \int_0^{\infty} S(\bar\omega)\, d\bar\omega - \int_0^{-\infty} S(\bar\omega)\, d\bar\omega \qquad (45)$$

Let $\bar\omega' = -\omega'$

$d\bar\omega' = -d\omega'$

$$\therefore\ \sigma_{12}^2 = \int_0^{\infty} S(\bar\omega)\, d\bar\omega + \int_0^{\infty} S(-\bar\omega')\, d\bar\omega' \qquad (46)$$

Since $S(\bar\omega) = A_1 + iB_1$

and $S(-\bar\omega) = A_1 - iB_1$ (47)

The imaginary parts of the two integrals cancel out, and only the real parts contribute to σ_{12}^2. Hence, Eqns. (43) and (44) differ only by a multiple; i.e,

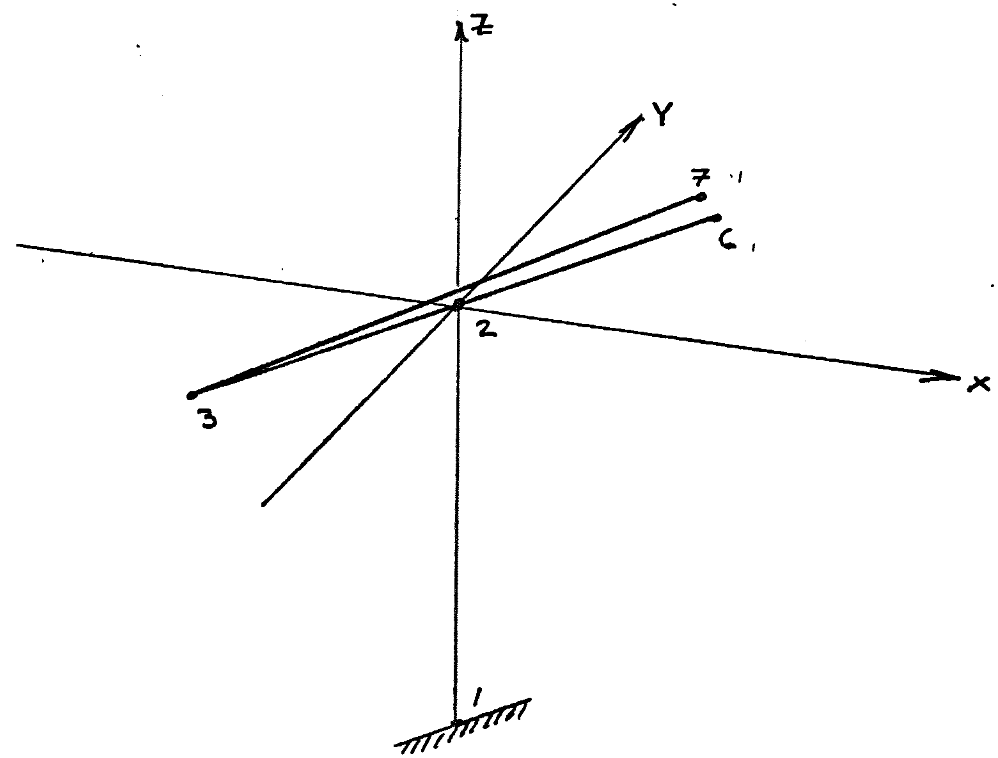
$$\boxed{\sigma_{12}^2 = \frac{\nabla_{kl}}{4}} \qquad (48)$$

i.e, the mean value of $\langle V_1 V_2 \rangle = \frac{1}{2}\sqrt{\nabla_{kl}}$ (49)

$$= \frac{1}{2}\sqrt{\sigma_{S_k}^2 - \sigma_{S_l}^2} \qquad (50)$$

and

$$S_{12}(\bar\omega) = \frac{S_k(\bar\omega) - S_l(\bar\omega)}{4} \qquad (51)$$



CANTILEVER BEAM MODEL

FIGURE 1

FREQUENCIES:

$f_1 = 13.38$ Hz

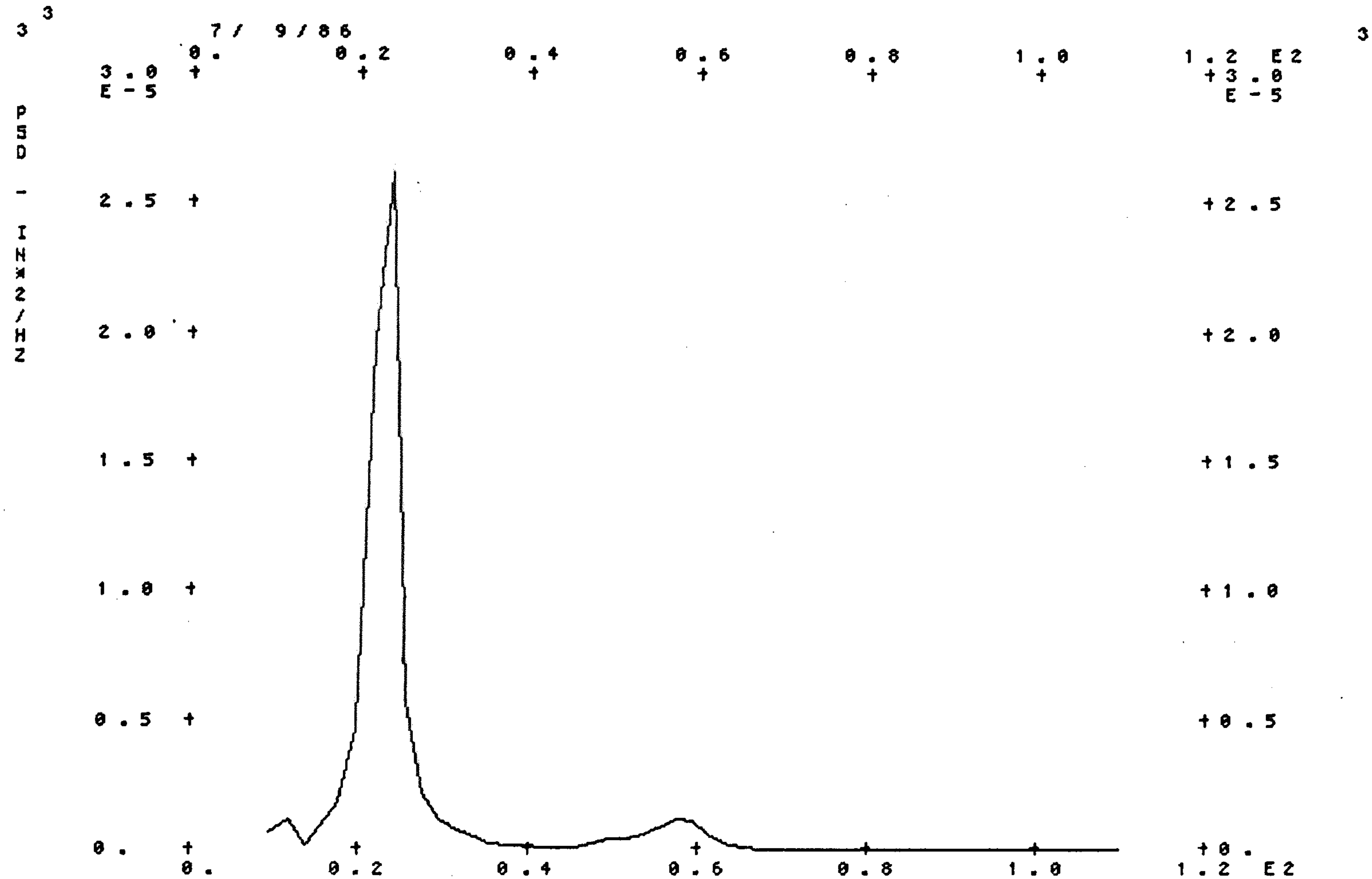
$f_2 = 13.81$ Hz

$f_3 = 23.17$ Hz

$f_4 = 35.17$ Hz

$f_5 = 48.18$ Hz

$f_6 = 59.33$ Hz



DISPLACEMENT OF POINT 22
MODAL -RANDOM TEST CASE
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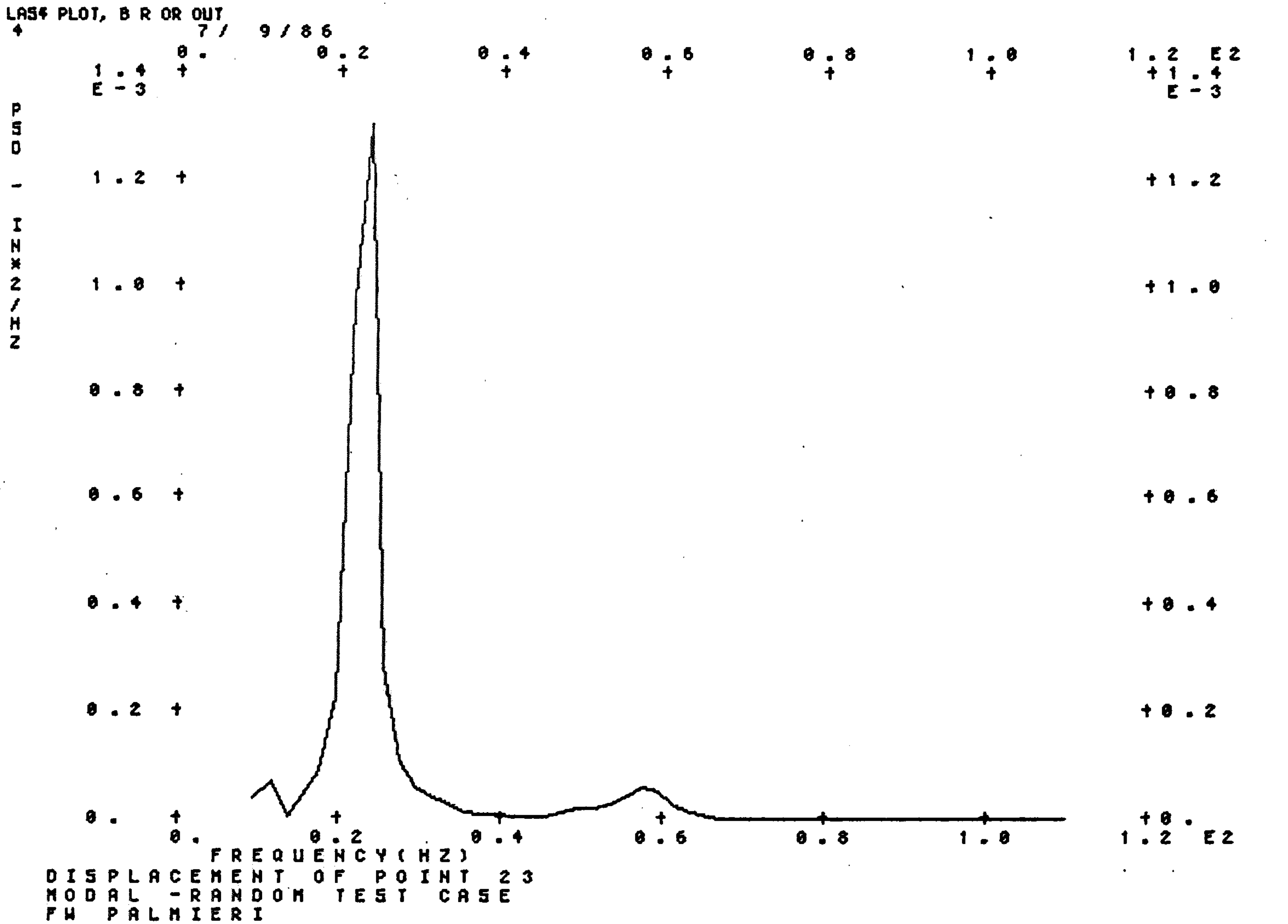
$\Delta X = \delta X_6 - \delta X_7$

$\Delta Y = \delta Y_6 - \delta Y_7$

$\Phi_{22} = \left[T(\Delta x + \Delta y)\right]^2 \Phi_i$

FIGURE 2



$\Delta X = \delta X_6 - \delta X_7$
$\Delta Y = \delta Y_6 - \delta Y_7$

FIGURE 3

$$\Phi_{23} = [T(\Delta x - \Delta y)]^2 \Phi_i$$