COMPUTATION OF THE EIGEN-FREQUENCIES OF ACOUSTIC CAVITIES: A NEW PENALTY METHOD

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ABSTRACT

A new penalty method for determining the natural frequencies and normal modes of acoustic cavities is presented. The acoustic medium is described in terms of the displacement field. A new easy-to-implement penalty function method is employed to enforce the irrotationality condition. This new method has significant advantages when compared to classical penalty function methods since it is more stable with respect to the penalty parameter.

INTRODUCTION

Analysis of acoustic cavities is a common problem in many automotive, aerospace, civil and nuclear engineering applications. The basic problem consists of determining the natural frequencies and normal modes of an acoustic medium of known geometry. In principle, there are two ways to approach this problem. One possibility is to describe the acoustic medium in terms of the pressure field. In this case, determination of the natural frequencies reduces to finding the eigenvalues of the well-known wave equation [1]:

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[1] Presently with Measurements Analysis Corporation, Torrance, Ca, USA.

$$\nabla^2 p = (\rho / B) \quad \partial^2 p / \partial t^2 \tag{1}$$

where ρ is the medium density and B is the Bulk modulus. With this formulation, the displacement field $u = (u_x, u_y, u_z)$ becomes a recovery variable that can be computed from:

$$-\text{grad } p = \rho \ \partial^2 u / \partial t^2 \tag{2}$$

A second alternative is to describe the acoustic medium in terms of the displacement field u. In this case, however, the irrotationality of the displacement field must be enforced. The pressure, p, can be recovered from

$$p = -B \ (\text{div } u) \tag{3}$$

This paper presents a new penalty function method to determine the natural frequencies and mode shapes of acoustic cavities. This approach uses the displacement field formulation and finite elements. For the sake of clarity, the method is presented for the case of classical acoustic cavities, i.e. cases in which the acoustic medium is confined to a volume defined by rigid walls or free boundaries.

STATEMENT OF THE PROBLEM

A more formal statement of the problem is needed at this stage. Consider a medium characterized by a known value of B and ρ and confined to a cavity of a given geometry. The strain energy of the system U is given by:

$$U = (1/2)\, B \int_v (\text{div } u)^2 dv \tag{4}$$

where V is the domain that defines the cavity. The kinetic energy T is given by

$$T = (1/2)\, \rho \int_v (du/dt,\, du/dt) dv \tag{5}$$

Upon appropriate finite element discretization the problem reduces to solving for the natural frequencies and normal modes of

$$-\omega^2[M]\{\phi\} + [K]\{\phi\} = \{0\} \tag{6}$$

subjected to

$$\text{curl } u = 0 \tag{7}$$

where $\{\phi\}$ is an eigenvector and ω represents the corresponding natural frequency. [M] and [K] are the mass and the stiffness matrices derived from Eqs. 4 and 5, respectively.

STANDARD PENALTY FUNCTION METHOD

The problem posed by Eq. 6 and subjected to the condition given by Eq. 7 can be reformulated using the so-called penalty function method. This idea, introduced first by Courant [2], was later brought by Babuska [3] into the finite element world. Zienkiewicz [4] and others [5, 6, 7, 8] have extensively applied this concept to a number of finite element problems. This approach redefines the strain energy term for the system to incorporate the penalty term -- a term that penalizes the violation of the condition that must be enforced. Thus, U becomes

$$U=(1/2)\,B\int_v (\text{div } u)^2\, dv + \alpha\,(1/2)\int_v (\text{curl } u)^2\, dv \qquad (8)$$

where α is a positive number, in general large compared to B. In this approach the stiffness matrix of the system can be decomposed into two terms: [K] and $[K_p]$. [K] is the contribution to the stiffness matrix resulting from the divergence operator, and $[K_p]$ represents the contribution to the stiffness matrix resulting from the curl operator. Therefore, the standard penalty function (PF) method leads to an eigenvalue problem of the form

$$-\omega^2[M]\{\phi\} + ([K]+\alpha[K_p])\{\phi\}=\{0\}. \qquad (9)$$

The problem with this formulation comes from the selection of α. A small α (small compared to B) will not adequately enforce the condition curl $u=0$. Ideally, a very large α would ensure satisfaction of curl $u=0$. In practice, however, an excessively large value of α produces serious numerical problems. This situation has been reported by several investigators [9, 10, 11]. One is therefore confronted with the problem of not knowing how to choose α; and what is worse, different α's can produce significantly different results. Therefore, the need for an improved technique is apparent.

MODIFIED PENALTY FUNCTION METHOD

The modified penalty function (MPF) method presented herein overcomes the deficiencies of the standard penalty method while maintaining its simplicity. The method is as follows:

STEP 1.

Solve for the eigenvectors of

$$-\omega^2[M]\{\phi\} + ([K]+\alpha[K_p])\{\phi\}=\{0\} \tag{10}$$

This step, which will give a first approximation of the true natural frequencies and normal modes, corresponds to the application of the standard penalty method as described in the previous section.

STEP 2.

Determine a modified stiffness matrix [K'] and a corresponding mass matrix [M'] defined as follows:

$$[K'] = [\Phi_q]^T[K][\Phi_q] \tag{11}$$

and

$$[M'] = [\Phi_q]^T [M][\Phi_q] \tag{12}$$

where $[\Phi_q]$ is the matrix of the first q eigenvectors produced in Step 1. (Note that [K'] excludes the penalty term).

STEP 3.

Solve for the eigenvectors and eigenvalues of the reduced system

$$-\omega^2 [M']\{\sigma\} + [K']\{\sigma\} = \{0\} \tag{13}$$

STEP 4.

An estimate of the natural frequencies of the actual system is given by the natural frequencies of the reduced system (Eq. 13); the desired eigenvectors (normal modes) are recovered using

$$\{\phi\} = [\Phi_q]\{\sigma\} \tag{14}$$

where $\{\sigma\}$ is a normal mode of the reduced system.

A large value of α in Step 1, will cause the natural frequencies of the system to be separated into two groups: (A) a first group of frequencies corresponding to curl-free eigenvectors; and (B) a second group of very high frequencies corresponding to eigenvectors that are not curl-free. Whether a certain vector $\{\gamma\}$ is curl-free or not can be determined simply by computing the ratio

$$\{\gamma\}^T[K_p]\{\gamma\}/\{\gamma\}^T[K]\{\gamma\} \qquad (15)$$

This number will be close to zero for the vectors of group (A); while for vectors of group (B) it will be much larger than zero. Ideally, in the reduction process (Step 2) one would choose q such that all vectors of group (A) are included.

In summary, the rationale behind this procedure is the following: first, a set of curl-free vectors is determined (Step 1); then, this set of vectors is used to reduce the size of the problem. The matrices that govern the reduced problem retain the dynamic characteristics of the original system, while producing curl-free solutions. This method, as will be seen in the subsequent examples, is extremely stable with respect to α.

The MPF method for acoustic analysis has been incorporated into a special version of MSC/NASTRAN, a general purpose finite element code. Extensive testing has been performed producing very encouraging results. Some examples are presented below.

EXAMPLES

Consider the acoustic cavity and its corresponding finite element discretization shown in Figure 1. In this example $B=100$ and $\rho=1$. All the walls are rigid, which means that the normal component of displacement at the wall must be zero. Table 1 shows the values obtained for the first two frequencies of the system using the standard penalty function method for different values of α. It can be seen that for small values of α ($\alpha=1$ and $\alpha=10$) the results deviate considerably from the true value. This is because the irrotationality condition is not appropriately enforced. For large values of α ($\alpha > 10^4$) the system locks producing, again, erroneous results. There is an intermediate region ($10^2 =< \alpha =< 10^3$) for which the results are acceptable. This situation -- which can be considered typical of the standard penalty method -- dramatically shows the importance of choosing *a priori* a correct value for α. Table 2 shows the results obtained for the same cavity using the MPF method and the same values of α. It can be seen that for small α's the results are still poor; but for large values of α the system is very stable; and the results are very accurate. This eliminates the burden of having to choose the right value of α *a priori*. Figure 2 depicts the typical behavior of the error for both methods. The advantages of the MPF method are apparent: it produces good results as long as α is large compared to B.

CONCLUDING REMARKS

A modified penalty function (MPF) method used to determine the natural frequencies and normal modes of acoustic cavities has been presented. This method has significant advantages when compared to standard penalty methods since it is extremely stable for large values of the penalty number. Therefore, it relieves the user from the burden (and risk) of having to guess the appropriate penalty coefficient. With the MPF method, a large α is sufficient to obtain good results; no numerical problems are encountered with large values of α.

Finally, the MPF method can be easily extended to treat any free vibration problem with constraints. An obvious application is the analysis of electromagnetic cavities. In this case, the stiffness matrix of the system results from the curl operator and the penalty term results from the divergence operator. In this context, the expression for U is

$$U = (1/2)\,\mu^{-1} \int_v (\text{curl } A)^2 dv + \alpha\,(1/2) \int_v (\text{div } A)^2 dv \qquad (16)$$

where μ is the permeability of the electromagnetic medium, A is the vector potential, and α is the penalty coefficient. The "mass matrix" results from the kinetic energy expression given by

$$T = (1/2)\, \epsilon \int_v (dA/dt, dA/dt)dv \tag{17}$$

where ϵ represents the permittivity of the medium. The similarity between this problem and the acoustic cavity problem makes the application of the MPF method straightforward.

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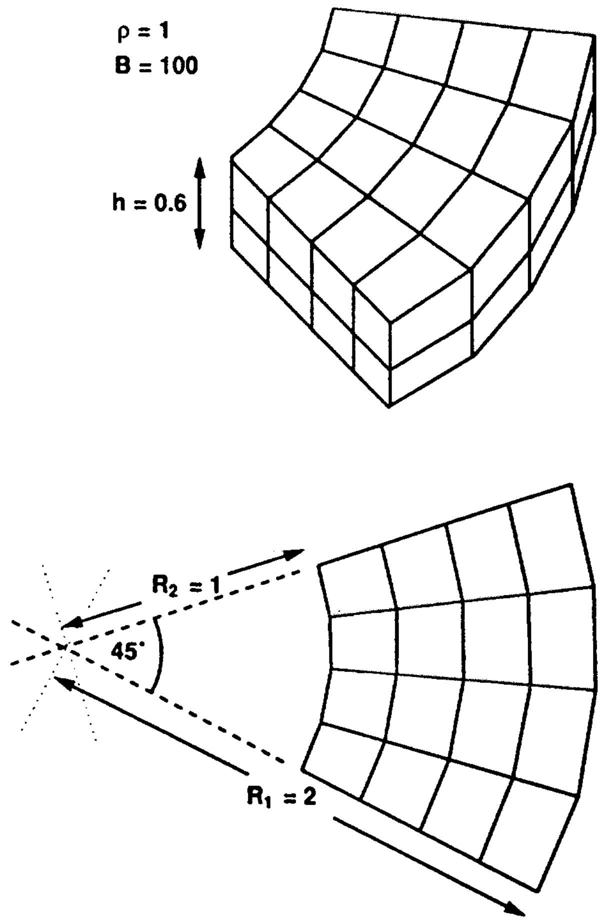

Figure 1. Acoustic Cavity, Closed Annular Segment.

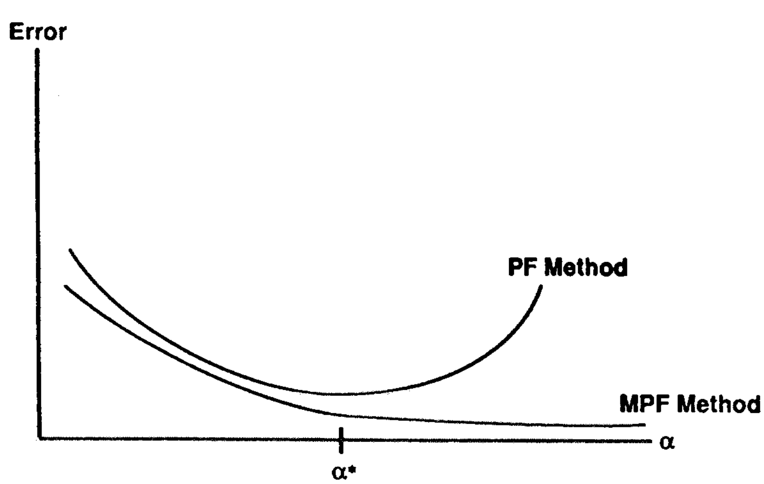

Figure 2. Behavior of the Error as a Function of α for the Standard Penalty Function (PF) Method and the Modified Penalty Function (MPF) Method.

TABLE 1. Natural Frequencies of a Closed Annular Segment Using the Standard Penalty Function Method.

Analytical Solution	Value of α, Penalty Coefficient						
	1	10	10^2	10^3	10^5	10^7	10^9
$f_1 = 4.11$	1.20	2.39	4.18	4.33	9.18	9.18	9.18
$f_2 = 5.08$	1.49	4.07	5.24	5.31	9.42	9.42	9.42

TABLE 2. Natural Frequencies of a Closed Annular Segment Using the Modified Penalty Function Method.

Analytical Solution	Value of α, Penalty Coefficient						
	1	10	10^2	10^3	10^5	10^7	10^9
$f_1 = 4.11$	.90	1.06	1.17	4.17	4.19	4.19	4.19
$f_2 = 5.08$	.95	2.15	4.17	5.23	5.23	5.23	5.23