A p-Version Element Error Indicator
for MSC/PROBE

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1. INTRODUCTION

One of the main advantages of using the p-version in MSC/PROBE is the ability to assess the quality of the results. This evaluation is performed through three quality control procedures:

1. relative error in energy norm;
2. equilibrium of elements and inter-element continuity;
3. continuity and convergence of point functionals.

The first criterion, using the global energy, is based on theoretical convergence rates and is easily evaluated. The third criterion, based on information at points, uses the expertise of the analyst to determine the quantities of interest in the problem.

The equilibrium of elements and inter-element continuity have been inconvenient to evaluate in the past, because they involve force resultants for each face of an element, as well as the faces of adjacent elements. An element error indicator, which is derived from these quantities, has been implemented in a prototype version of MSC/PROBE. This error indicator reliably combines the equilibrium and continuity information into one quantity for each element, which can then be viewed as color-coded elements and used to determine the need for increasing the polynomial level or refining the mesh.

In this paper, the background of the p-version and the calculation of the p-version element error indicator will be discussed. The application and performance of the indicator will be demonstrated through examples. Related issues, such as the use of the indicator in an adaptive manner, will be explored.

2. P-VERSION OF THE FINITE ELEMENT METHOD

2.1 h-Version and p-Version

In the h-version of the finite element method, the number of shape functions is fixed for each element. The shape functions are usually polynomials of low degree, normally 1 or 2. These correspond to the linear and isoparametric elements in traditional finite element programs. The error of approximation is controlled by mesh refinement. Since the characteristic dimension of the elements is often denoted by h, this is called the h-version.

In the p-version of the finite element method, the number of shape functions for each element can be increased to include polynomials of higher degree. In this case, the error of approximation is controlled not only by mesh refinement, but also by increasing the polynomial degree of elements. Since the polynomial degree of the element is denoted by p, this is called the p-version.

Structural analysis can generally be divided into two areas: global analysis, where the general behavior, such as load paths, of a structure is important; and local analysis, where the specific response, such as the detailed stress state, of a single component is important. The former is well-suited to the h-version, which is used in MSC/NASTRAN, while the latter can take advantage of the p-version, which is incorporated in MSC/PROBE.

2.2 Extension Processes

Extensions involve increasing the number of degrees of freedom in an orderly fashion, so as to approach the exact solution as the number of degrees of freedom becomes infinite. Extension processes are important, because both the accuracy of the approximation and the control of the error are based on extensions. There are three basic types of extensions, corresponding to the h-version, p-version, and a combination of the two.

The global quality of the approximation is typically measured by the relative error in energy norm, which is related to the root-mean-square error in stresses, and so the performance of the extensions can be measured by the rate of convergence of the relative error with respect to the number of degrees of freedom.

It has been shown theoretically that convergence to the exact solution occurs for all extension processes, but with the differences occurring in the rates of convergence. These theoretical rates of convergence are shown in Figure 1.

The h-extension with uniform mesh refinement is the least efficient extension process, and for problems with singularities the rate is dependent on the polynomial level and strength of the singularity.

The p-extension with an ungraded mesh for problems with singularities has a convergence rate twice as great. Typically neither of the latter two extension processes would be used in a real problem.

The h-extension with optimally graded meshes, where optimal is defined as each element having the same error, has a convergence rate dependent only on the p-level. For smooth problems, the optimal mesh is a uniform mesh, but otherwise the optimal mesh is difficult to create in practice.

The p-extension with strongly-graded meshes around the singularities, which is how meshes for MSC/PROBE should be designed, consists of two parts: first is the exponential convergence rate, which approaches the theoretically optimal convergence rate; second is the algebraic rate, which is parallel to the p-extension for an ungraded mesh. More graded elements would postpone the transition, and the goal is to use enough grading to get the desired accuracy before the transition.

The hp-extension has a completely exponential convergence rate and is the most efficient. This is also the same rate as p-extension for smooth problems.

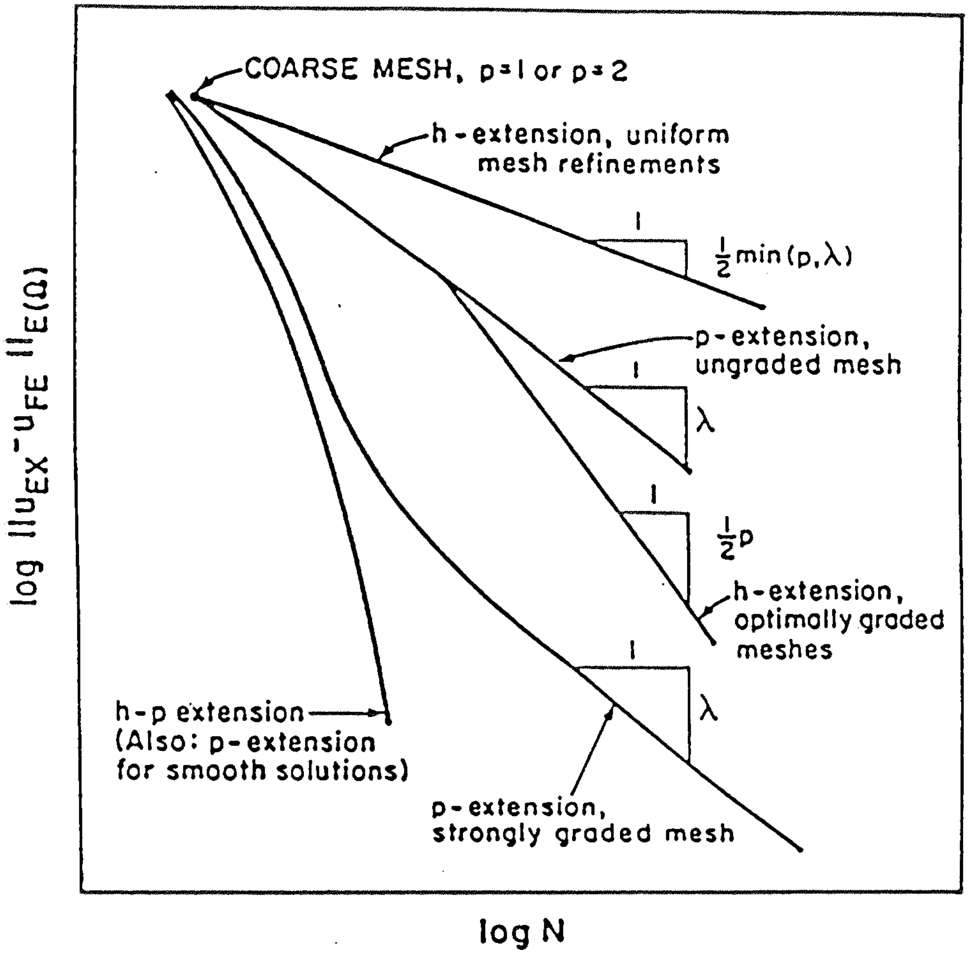

Figure 1: Theoretical Rates of Convergence

2.3 Quality Control Procedures in the p-Version

After the p-extension has been performed, the quality and reliability of the final solution must be assessed. This involves three quality control procedures:

1. relative error in energy norm,
2. equilibrium of elements and inter-element continuity,
3. continuity and convergence of point functionals.

The first quality control procedure is the relative error in energy norm, which is based on an estimated global strain energy computed from theoretical considerations. The estimate uses a sequence of three solutions which must be hierarchic, so that the lower-order solutions are embedded in the higher-order solutions. Since the polynomial shape functions in MSC/PROBE were developed to be hierarchic, this condition is met. The global energy is the only quantity for which theoretical estimates exist, and provides a global measure of the solution quality. After the solution is verified to be good in a global sense, more localized checking is then performed.

The second quality control procedure is the equilibrium of elements and inter-element continuity. This involves explicitly integrating the stresses over the element boundaries to calculate the resultant forces. The forces for each side of an element may then be summed to get the resultant force on the element, which should be zero. Similarly, the forces between two adjacent elements may be compared to check whether Newton's action/reaction principle is satisfied. If these element-level tests, performed on elements in the area of interest, are satisfied, then even more localized checking is performed.

The third quality control procedure is the continuity and convergence of point functionals, such as stresses and strains. This is the most localized and rigorous test, since the previous tests involved integrations over larger areas that could hide local problems. This test involves comparing the continuity of point functionals on boundaries between adjacent elements in the region of interest. It also involves examining the convergence of the functionals at any particular point as the polynomial level is increasing. This can not only give a value of the function, but also an estimate of the bounds on the accuracy.

Once all three quality control procedures have passed, the finite element approximation may be accepted. Then the quantities of interest, such as the location and magnitude of the maximum stress, may be calculated from the solution and used in making engineering decisions.

3. P-VERSION ELEMENT ERROR INDICATORS

The purpose of creating a p-version element error indicator for MSC/PROBE is to facilitate the second quality control procedure, which is the equilibrium of elements and inter-element continuity. The equilibrium of each element is easily compared, but the inter-element continuity requires comparison of values from adjacent elements or from boundary conditions. In three dimensions with complex geometry, this can become a cumbersome process, which would tempt the analyst to skip this quality assurance step.

The element error indicator reliably combines the equilibrium information from the specified element and the continuity information from the adjacent elements and boundary conditions. The results is then a single number which can be viewed as color-coded elements in a plot, rather than tables of numbers.

The indicators can be compared for a given p-level, so that decisions regarding increasing the polynomial level or refining the mesh can be made. They can also be compared for different p-levels on a specific element, so that the effect of increasing the p-level can be evaluated.

The interpretation of the error indicators uses the expertise of the analyst; it is not intended to be a "black box" approach. The engineer must remain an integral part of the analysis process. The problem being analyzed can conceptually be divided by the analyst into two regions: the region of primary interest, and the region of secondary interest. The region of primary interest is where the detailed solution is sought, whereas the region of secondary interest is only modelled to carry the load and provide appropriate boundary conditions for the region of primary interest. The p-level could be limited in the secondary regions, which is a capability currently offered in MSC/PROBE.

The error indicators must be evaluated with these two regions in mind. If the indicators are high in the secondary area, it might not be necessary to increase the p-level or refine the mesh, depending on the proximity of the primary area; this is a decision which should be made by the analyst, not by the program. Having an automatic procedure which tries to decrease the error indicators without this knowledge would be wasteful. If the indicators are high in the primary area, then the p-level should be increased or the mesh should be refined. Theoretically, for an optimal mesh, each element should contribute equally to the error. This could be done by refining elements with larger indicators and increasing the p-level over the whole primary region to decrease the total error.

After the indicators have been reduced to a satisfactory level in the primary region, the third quality assurance step, continuity and convergence of point functionals, should be checked. The use of the error indicators does not replace this third step, which is the most localized test.

The difference between error estimators and error indicators should be noted. Error estimators provide a numerical value for the error of some quantity, and their accuracy is measured by a quantity called the effectivity. Error indicators provide a numerical value which must be compared with other values to determine where the error is highest; they do not provide a specific value for the error. Since only the relative values of the indicators are important, they may be normalized with some convenient values. The relative error in energy norm computed in MSC/PROBE is an error estimate; it provides a value for the error, which is based on theoretical convergence rates, and has a high effectivity without any adjustment factors. Because MSC/PROBE already has an estimator for the global solution, the element error indicators do not have to serve as estimators. For other programs, it would be necessary for such element error indicators to serve as estimators, since they have no estimators. There are some other element quantities which claim to be estimators, but these generally have low effectivities over a wide range of models.

4. APPLICATION OF THE ELEMENT ERROR INDICATORS

The first model to illustrate the properties of the error indicators is a three-element tension strip. This problem is a smooth problem, which means that there are no stress singularities, although there is a stress concentration. The mesh is shown in Figure 2, and the behavior of the indicators as the p-level increases is shown in Figure 3, where the logarithm of the indicators is plotted.

It can be seen that the indicators decrease very rapidly with the p-level. The indicators for elements 1 and 2 are approximately equal at each p-level, which is expected, since those two elements are similar. The indicator for element 3 is consistently lower than those for elements 1 and 2, which is also expected, since that element is not curved and primarily carries the load. It would have been possible to use a lower p-level for element 3.

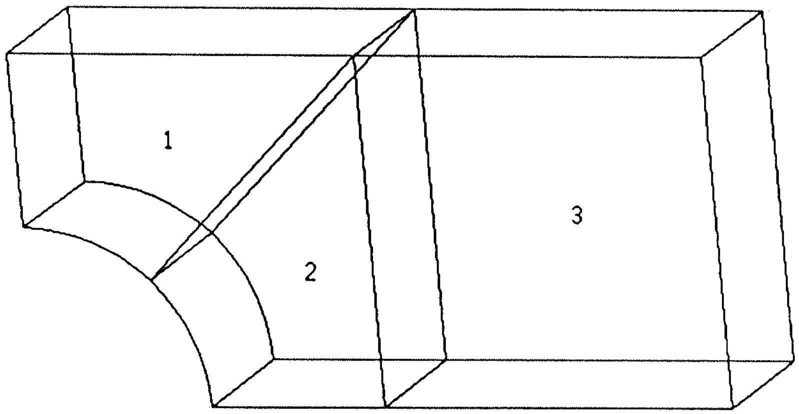

Figure 2: Three-Element Tension Strip

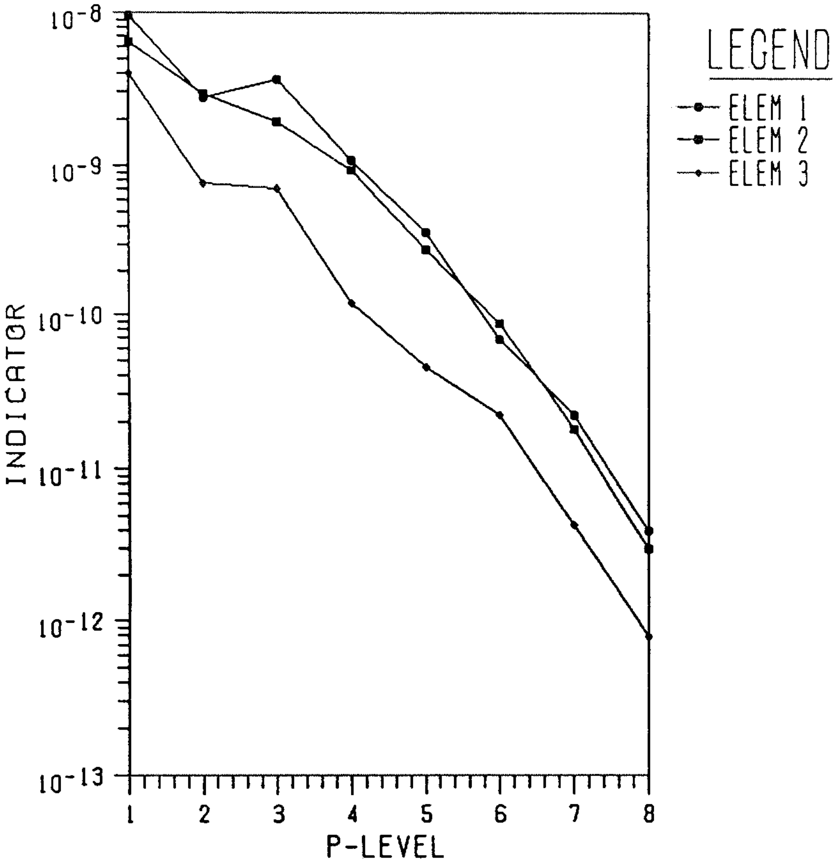

Figure 3: Error Indicators for Tension Strip

The second model to illustrate the error indicators is the 18-element splicing fixture. This model is also smooth. The mesh is shown in Figure 4, and the decrease of the indicators with p-level for all the elements near the hole is shown in Figure 5.

As for the tension strip, the indicators decrease rapidly with the p-level. Note that the behavior is not monotonic; since the indicators are computed from stress functions, as opposed to energy functions, monotonic behavior is not required. There is some different behavior at even and odd p-levels as the different components of the indicator are excited. Some of the elements in the fillet area of the fixture consistently have the highest values. These elements are highly distorted and are subject to the transition of the load between membrane and bending action; however, they are away from the area of primary interest, which is the inside of the hole.

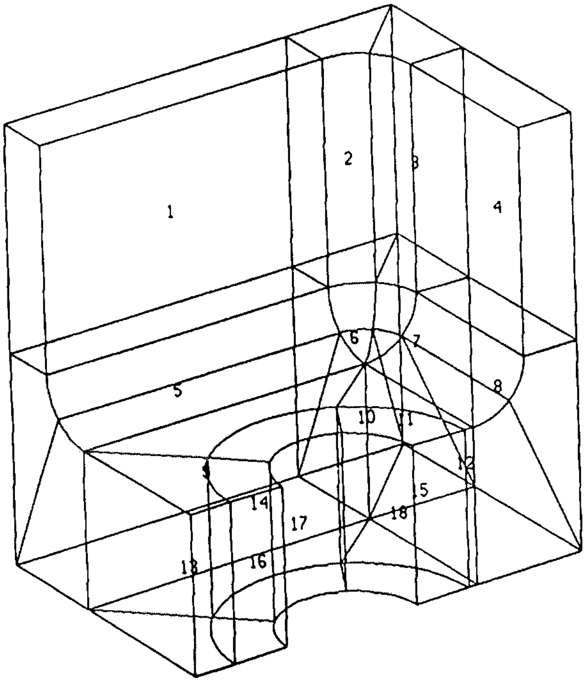

Figure 4: 18-Element Splicing Fixture

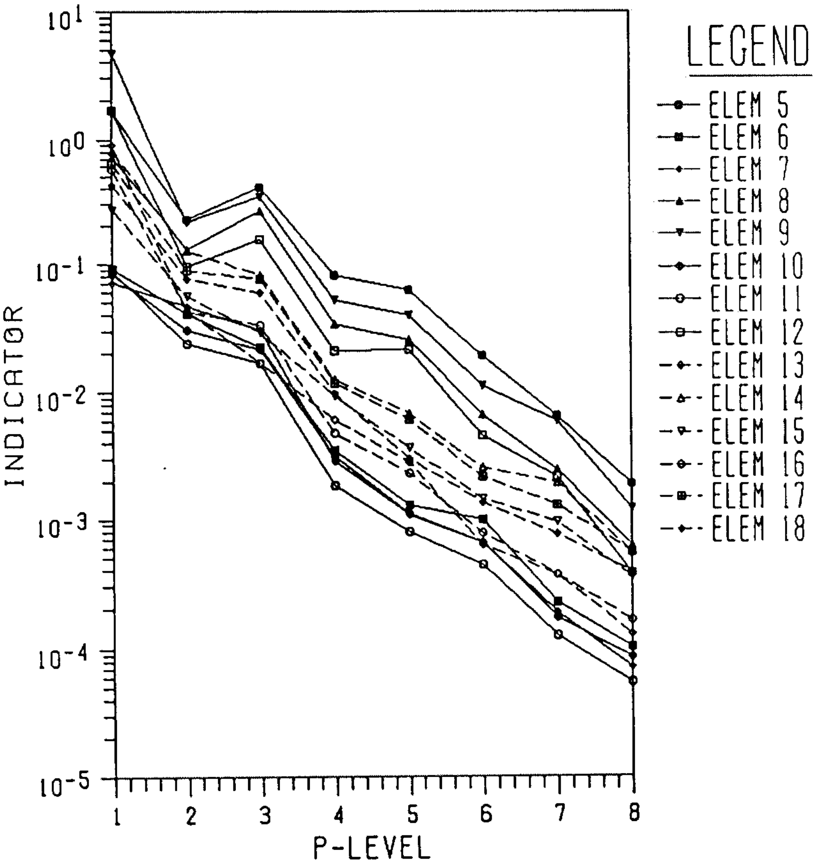

Figure 5: Error Indicators for Splicing Fixture

Some parametric studies were conducted to evaluate the behavior of the indicators seen in the previous models. A short, straight cantilever beam of unit dimensions was used so that the singularities at the fixed end would dominate the whole model. Several meshes were used with grading ratios of 1/2, 1/5, and 1/10 towards the fixed face. Straight-sided hexahedra were used, so that distortion would not be a problem, although aspect ratios could be. Meshes with one to four elements were used for each of the grading ratios.

Figure 6 shows the behavior of the indicators as the p-level increases for the three two-element meshes. The indicators for the 1/2 grading ratio show some erratic behavior. That is because the mesh is uniform, in spite of the singularities. The erratic behavior shows that the problem is being meshed incorrectly, so that the solution is poor even at high p-levels. The indicators for the 1/10 grading ratio are more well-behaved, indicating that a better mesh is being used. The indicators for these two elements are also close in value, which, as was stated earlier, is the optimal distribution of the error.

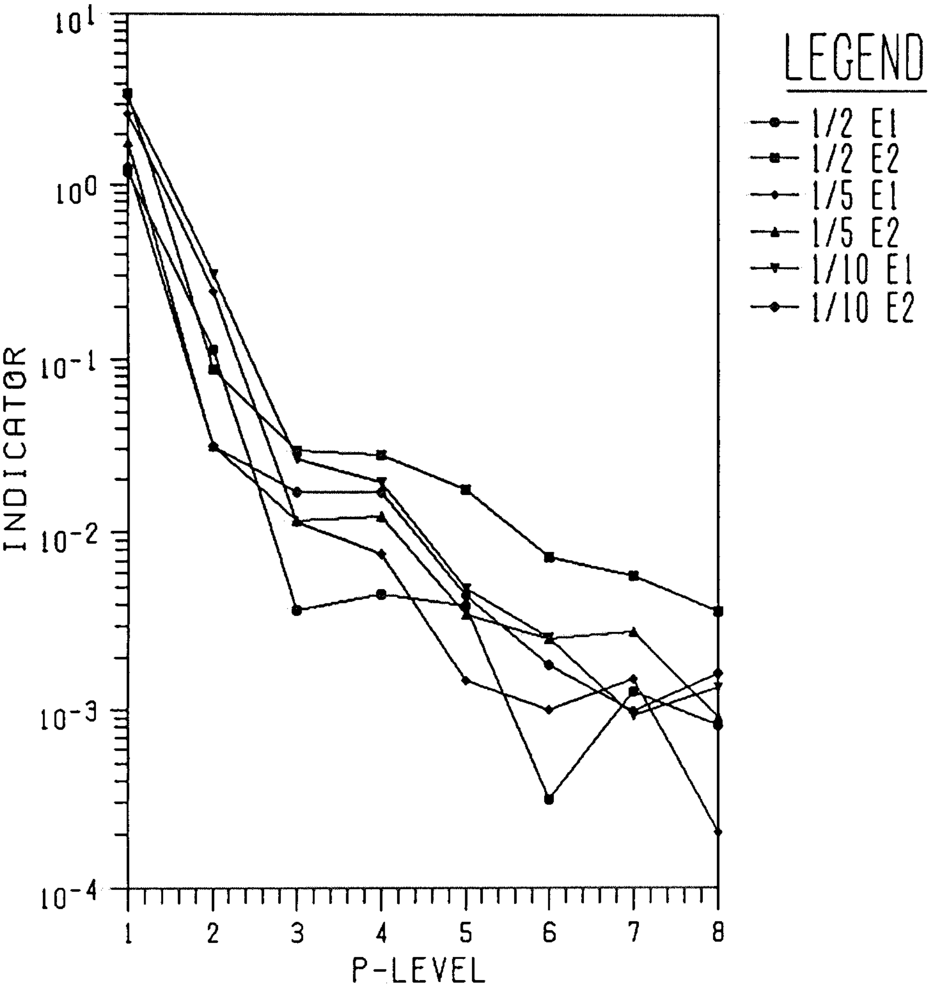

Figure 6: Indicators for Two-Element Beams

Figure 7 shows the behavior of the indicators as the p-level increases for the largest element of the four meshes for the 1/5 grading ratio. It can be seen that the indicators decreased significantly when the second element was added, which isolated the first element from the singularity. The addition of the third element changed the indicators for the first element only slightly, not even consistently improving them over all the p-levels, and the addition of the fourth element had negligible effect on the indicators for the first element.

The 1/5 grading ratio was used for this study, because it is close to the theoretically optimal grading ratio. The values of the indicators show that for this ratio, adding an element next to the singularities effectively isolates the singularities from the other elements. If results were sought inside the second element, then the third element would have to be added with the same grading ratio. The 1/2 grading ratio did not effectively isolate the singularities, such that the indicators for the first element continued to decrease as the third and fourth elements were added.

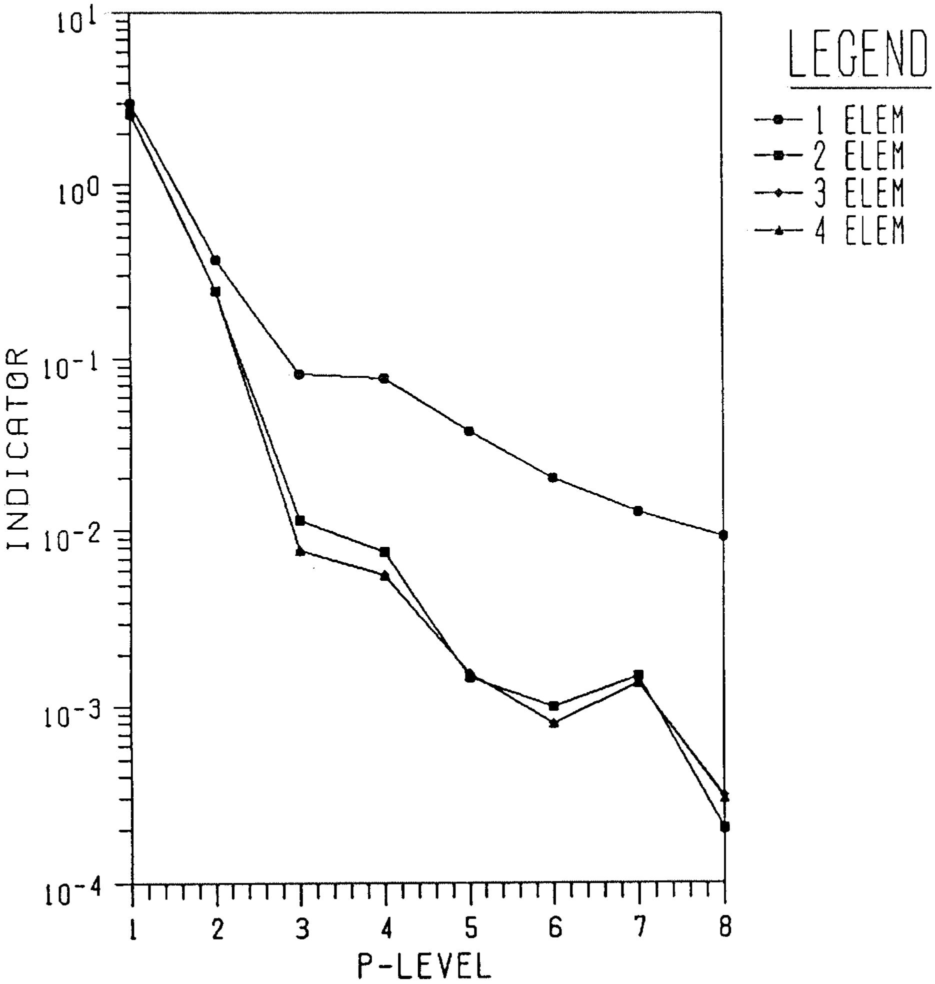

Figure 7: Indicators for Largest Element on 1/5 Beams

The performance of the indicators was also tested on some larger problems, which are more typical of user applications. The first model was a section of an automobile crankshaft subjected to a large moment, and the goal of the analysis was to find the maximum stress. The mesh had 107 elements and is shown in Figure 8, with the elements having the maximum values of the indicators highlighted.

The elements having the highest indicators are large elements which occur at the point of load application on the main journal and carry the moment into the crank cheek. Since the elements are large and are carrying a large part of the load, it is expected that they would have higher indicators. The elements in the two fillets, which are the critical elements, are smaller and have no singularities, so that the solution is smooth in those regions, even though the stresses are high. The highest stresses occur in the fillet between the crank cheek and the crank pin, which is opposite the point of load application. Therefore no refinement is necessary on the journal, and the higher indicators can be ignored. In addition, much of the crank cheek has very low indicators and could have used a lower p-level.

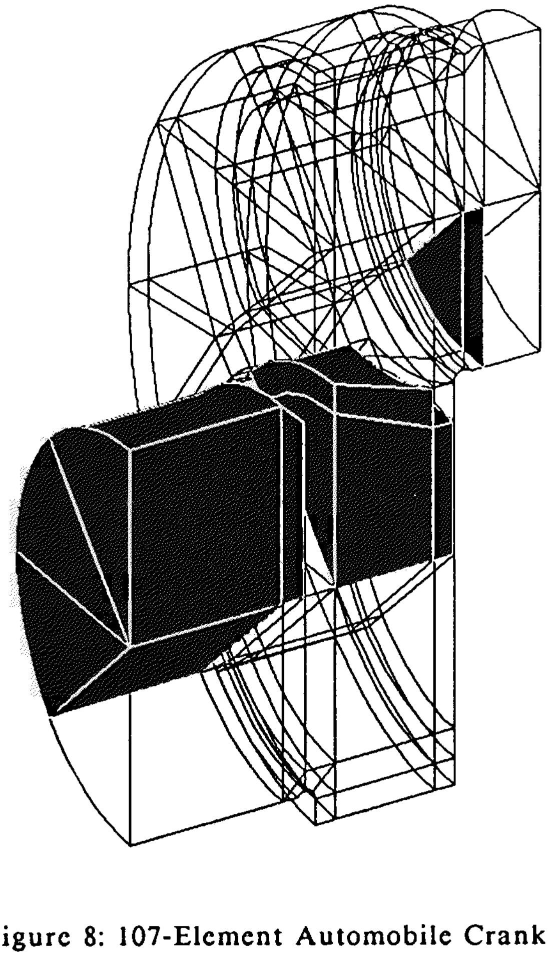

Figure 8: 107-Element Automobile Crankshaft

The second large model was an automobile compressor bracket subject to an eccentric loading through the two lugs to simulate a belt on a shaft. The four holes on the base were constrained to simulate bolts. The mesh had 325 elements and is shown in Figure 9, with the elements having the maximum values of the indicators highlighted.

The elements having the highest indicators occurred on the two ribs leading from the base to the holes in the lugs. This ribs carry most of the load from the holes to the base, since the surrounding area is thinner to minimize the weight. In addition, the fillets around the ribs were not modelled, since that area was of secondary interest, so that there are singularities there. The region of primary interest was the base with the four bolt holes. There are some elements around the bolt holes that have high indicators, as shown in the figure. These elements would have to be carefully examined to see if the solution is converging within them. The indicators for these elements were significantly lower than those for the ribs, and the solution was shown to converge around the holes, so that further refinement was not necessary.

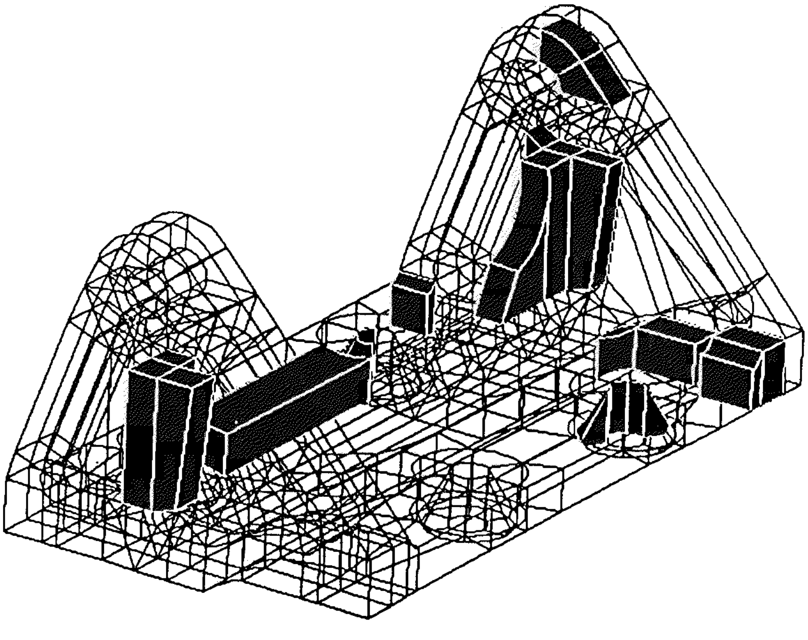

Figure 9: 325-Element Compressor Bracket

5. ADAPTIVITY WITH MSC/PROBE

MSC/PROBE currently offers guided h- and p-adaptivity. There is an orderly sequence of three quality control procedures, as described previously, which enables the analyst to verify the accuracy of the solution. This approach allows the engineer to use his expertise in deciding what is and what is not important, relative to the goals of the analysis, and therefore can prevent unnecessary effort in reducing errors in irrelevant areas of the model.

There are currently some commercial programs which have automatic h- or p-adaptivity. These programs will automatically run the problem, compute some measures of the error, refine the mesh or increase the p-level with little or no user guidance, and then repeat the process until some measure has converged.

For h-version adaptivity, the error indicator is typically based on nodal stress discontinuities, which assumes that a continuous stress field is the most accurate. The programs refine the mesh in areas with high indicators, reduce the global mesh size to avoid bad transitions, and use special mesh techniques at singularities if convergence is slow. Each step requires that the model be remeshed and the problem be resolved. This method does not have an orderly progression of solutions, since a new mesh is used, and it may refine in irrelevant areas. In addition, the error in energy is not necessarily related to the error in stresses.

For p-version adaptivity in MSC/PROBE, the error estimator is based on convergence of strain energy, and the error indicator is based on integrations of stresses, which assumes equilibrium must be satisfied. The final test is based on the convergence of the quantity of interest. The analyst can select areas of interest and increase the p-level selectively with or without mesh refinement, based on the error indicators. This method does have an orderly progression of solutions, and often no mesh refinement is necessary. It takes advantage of the analyst's expertise, but still provides enough feedback for the novice user, and prevents refinement in irrelevant areas. The final test is on the quantity of interest, instead of energy. The result of this method is a more conservative approach.

While MSC/PROBE is not automatically adaptive in the sense of other programs, an intelligently designed mesh by a trained user often yields sufficient accuracy by increasing the p-level without any mesh refinement. This fact, coupled with the other well-known advantages of the p-version, means that the analysis can be completed more efficiently.

6. SUMMARY

The p-version of the finite element method and its implementation in MSC/PROBE were presented, and rates of convergence for the three basic extension processes were compared.

Three quality control procedures for the p-version available in MSC/PROBE were presented. These are:

1. relative error in energy norm,
2. equilibrium of elements and inter-element continuity,
3. continuity and convergence of point functionals.

After the solution has been calculated, the quality control procedures are applied to assess the accuracy of the solution. With multiple solutions on the same mesh, convergence of the quantity of interest may be observed.

The calculation of a p-version element error indicator for MSC/PROBE, which reliably combines the element equilibrium and continuity information into a single number, was presented. The evaluation of these indicators takes advantage of the expertise of the analyst instead of using a "black box" approach. The indicators guide the analyst to regions of error in the problem, where the p-level may be increased or the mesh may be refined, if the region is of interest.

The element error indicators were applied to three sets of problems. The first set showed the basic behavior of the indicators for smaller problems. The second set performed some parametric studies to explain some of the behavior evident in the first set. The third set consisted of larger problems, which are more typical of user applications. In all three sets, the indicators behaved as expected.

Adaptivity with MSC/PROBE was discussed and compared with some programs which have automatic h- and p-adaptivity. The approach in MSC/PROBE is to enable the analyst to verify the accuracy of the solution with the orderly sequence of quality control procedures. This approach is more conservative and takes advantage of the analyst's experience, but also provides enough feedback for the novice. Having the analyst evaluate what is and what is not important, relative to the goals of the analysis, saves unnecessary effort in reducing errors and therefore allows the analysis to be completed more efficiently.

7. BIBLIOGRAPHY

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