EXACT CALCULATION OF MINIMUM MARGIN OF SAFETY FOR FREQUENCY RESPONSE ANALYSIS STRESS RESULTS USING YIELDING OR FAILURE THEORIES

François Charron
Université de Sherbrooke
Sherbrooke, Québec, Canada

Louis Donato and Mark Fontaine
Spar Aerospace Limited
Ste-Anne-de-Bellevue, Québec, Canada

ABSTRACT

In static analysis, the calculation of minimum margins of safety using yielding (Von Mises,...) or failure theories (instability for honeycomb structure,...) requires all stress components (3D case : $\sigma_{xx},\sigma_{yy},\sigma_{zz},\tau_{xy},\tau_{xz},\tau_{yz}$; both magnitude and sign) for a specific element. In frequency response analysis, the stress component magnitude and sign are function of the reference phase angle and the phase angle of each of the various stress components. When the phase angle difference between the various stress components is almost equal to 0 or 180 degrees, the calculation of the minimum margin of safety is simple. However, in the general case, the minimum margin of safety will be dependant upon both the reference phase angle as well as the phase angle of each various stress components. This paper describes a method used for the calculation of the **exact** minimum margin of safety for the general case. For the 2D and 1D elements, the exact minimum margin of safety is evaluated at the lower and upper fibers of the element where the flexural stress is maximum and the transverse shear stress contribution is equal to zero. The calculation of the exact minimum margin of safety is done by a general stress processor using the MSC/NASTRAN OUTPUT2 file.

1.0 INTRODUCTION

The final dimensionning of aerospace structure components is based on detailed stress analysis. It is important to precisely calculate the margin of safety using various yielding, failure or instability criteria. The precise calculation of the margin of safety for complicated stress fields using these criteria requires all stress components (3D case : $\sigma_{xx},\sigma_{yy},\sigma_{zz},\tau_{xy},\tau_{xz},\tau_{yz}$; magnitude and sign) for a specific element.

In static analysis, all stress components ($\sigma_{xx},\sigma_{yy},\sigma_{zz},\tau_{xy},\tau_{xz},\tau_{yz}$), principal stresses ($\sigma_1,\sigma_2,\sigma_3$) and the equivalent stress (σ_e, Von Mises) are available to the analyst from a finite element program like MSC/NASTRAN. In this case, the calculation of the margins of safety for ductile materials is straightforward because the equivalent stress using the Von Mises theory is provided by the FEM software. Unfortunately, the same calculation for a honeycomb panel using a local instability failure criteria, such as local crippling requires the use of all stress components and the effort required to compute the margins of safety for a complex structure is significant.

In frequency response analysis, all stress components are complex numbers. A particular stress component may be expressed as follows:

$$\sigma_{xx}(t) = |\sigma_{xx}| \cdot \sin(\omega t + \beta) \tag{1}$$

where $\sigma_{xx}(t)$	:	normal stress component in x direction
$|\sigma_{xx}|$	:	normal stress magnitude for σ_{xx}
β	:	normal stress phase angle for σ_{xx}
ω	:	excitation frequency (rad/sec)
t	:	time
ωt	:	reference phase angle

Typically, all the stress components will have different phase angles. When the phase angle difference between the various stress components is almost equal to 0 or 180 degrees, a good approximation can be obtained by simply using a consistent sign convention. When the phase angle difference between the various stress components is significantly different from 0 or 180 degrees (e.g excitation frequency in the viscinity of closely spaced modes), the reference angle (ωt) for which the margin of safety will be minimum is difficult to evaluate.

1.0 INTRODUCTION (...)

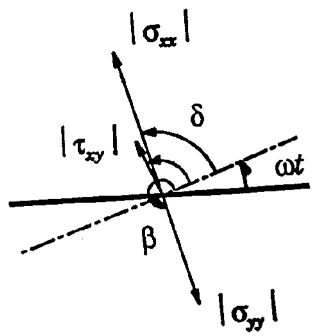
The aim of this paper is to present a technique for the calculation of the minimum margin of safety using stress results obtained from either a static or a frequency response analysis.

2.0 PROBLEM DEFINITION

The best way to define and to illustrate the problem is through two simple examples using a stress field obtained from a 2D element (e.g. CTRIA3 or CQUAD4) and the Von Mises criterion.

2.1 Example no 1

In this example the phase angle difference for the various stress components is almost 0 or 180 degrees. The following figure shows the three stress components with respect to a given the reference phase angle.



$|\sigma_{xx}|$ = *20000 PSI*
$|\sigma_{yy}|$ = *15000 PSI*
$|\tau_{xy}|$ = *8000 PSI*
δ = *85°*
β = *266°*
ϕ = *92°*

FIGURE 1 2D stress component vectors with respect to the reference phase angle (ωt)

From this figure, the analyst may use one of the following four options for the calculation of the margin of safety:

i) To project σ_{xx} and σ_{yy} vectors on τ_{xy} vector,
ii) To project σ_{xx} and τ_{xy} vectors on σ_{yy} vector,
iii) To project τ_{xy} and σ_{yy} vectors on σ_{xx} vector,
iv) To use directly the value of $|\sigma_{xx}|$, $|\sigma_{yy}|$ and $|\tau_{xy}|$ and to assign a positive sign to σ_{xx} and τ_{xy} and a negative sign to $|\sigma_{yy}|$.

The table on the following page summarizes the results obtained from these four options and compares them to the exact solution.

2.0 PROBLEM DEFINITION (...)

CASE	DESCRIPTION	σ_{xx} (PSI)	σ_{yy} (PSI)	τ_{xy} (PSI)	σ_e (PSI)	MARGIN(1) OF SAFETY
1	OPTION I	19851	-14918	8000	33238	20.4%
2	OPTION II	-19997	15000	-7956	33388	19.8%
3	OPTION III	20000	-14998	7940	33377	19.8%
4	OPTION IV	20000	-15000	8000	33422	19.7%
5	EXACT	19782	-14872	7997	33144	20.7%

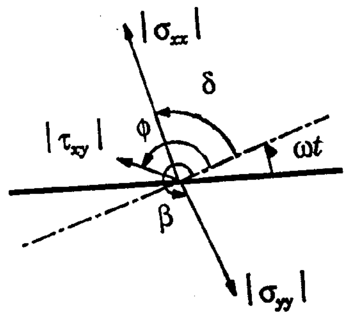
(1) σ_{yield} = 40000 PSI
(2) Margin of safety = $((\sigma_{yield} / \sigma_e)-1)\cdot 100$

TABLE 1 Equivalent stress (Von Mises) and margin of safety for various cases

From table 1, we can conclude that when the phase angle difference between the stress components is close to either 0 or 180 degrees, it is possible to use one of the above listed options to obtain a good approximation of the equivalent Von Mises stress and consequently, the margin of safety.

2.2 Example no 2

In this example, the phase angle difference between the various stress components is general. This situation arises when the excitation frequency is in the viscinity of closely spaced modes or when the modal density is high. The following figure shows the three stress components with respect to a given the reference phase angle.



$|\sigma_{xx}|$ = *20000 PSI*
$|\sigma_{yy}|$ = *15000 PSI*
$|\tau_{xy}|$ = *8000 PSI*
δ = *85°*
β = *275°*
ϕ = *130°*

FIGURE 2 2D stress component vectors with respect to the reference phase angle (ωt)

2.0 PROBLEM DEFINITION (...)

Using the four options presented in example 1, the following results are obtained.

CASE	DESCRIPTION	σ_{xx} (PSI)	σ_{yy} (PSI)	τ_{xy} (PSI)	σ_e (PSI)	MARGIN[1] OF SAFETY
1	OPTION I	14142	-12287	8000	26771	49.4
2	OPTION II	-19696	15000	-6553	32205	24.2
3	OPTION III	20000	-14772	5656	31774	25.9
4	OPTION IV	20000	-15000	8000	33421	19.7
5	EXACT	19958	-14572	5279	31386	27.5

(1) σ_{yield} = 40000 PSI

TABLE 2 Equivalent stress (Von Mises) and margin of safety for various cases

From table 2, it may be seen that there is a significant scatter between the margins of safety calculated using the four options. These differences can lead to either an over or under estimation of the actual margin of safety.

3.0 THEORY

In order to address the difficulties brought out in example 2, a method to calculate the minimum margin of safety for various yielding, failure and instability criteria was developped. In the next section, a detailed description is given for the method applied to the yielding criterion. For this case, an analytical expression is derived. For the other criteria, no closed form solution exist and the problem is solved numerically for each element.

3.1 YIELDING CRITERION - DUCTILE MATERIAL

The stresses resulting in a 2D element from a harmonic excitation are as follows:

$$\sigma_{xx}(t) = |\sigma_{xx}| \cdot \cos(\omega t + \delta) \tag{2}$$

$$\sigma_{yy}(t) = |\sigma_{yy}| \cdot \cos(\omega t + \beta) \tag{3}$$

$$\tau_{xy}(t) = |\tau_{xy}| \cdot \cos(\omega t + \phi) \tag{4}$$

3.1 YIELDING CRITERION - DUCTILE MATERIAL (...)

If we let the 'ωt' equal θ in equations 2 thru 4 and introduce these new relationships in the general 2D Von Mises expression [3], it is possible to demonstrate that the equivalent stress is as follows:

$$\sigma_e^2(\theta) = (|\sigma_{xx}|^2 \cdot \cos^2(\theta+\delta) + |\sigma_{yy}|^2 \cdot \cos^2(\theta+\beta) + 3 \cdot |\tau_{xy}|^2 \cdot \cos^2(\theta+\phi) - |\sigma_{xx}|\,|\sigma_{yy}| \cdot \cos(\theta+\delta) \cdot \cos(\theta+\beta)) \tag{5}$$

Equation 5 can be simplified to the following form:

$$\begin{gathered} \sigma_e^2(\theta) = H(\theta) \\ \sigma_e = \sqrt{H(\theta)} \end{gathered} \tag{6}$$

If we set the derivative of equation 6 with respect to θ equal to zero,

$$\frac{d\sigma_e(\theta)}{d\theta} = \frac{1}{2 \cdot \sqrt{\mathrm{H}(\theta)}} \cdot \frac{d\mathrm{H}(\theta)}{d(\theta)} = 0 \tag{7}$$

and using this expression, it is possible to show that the reference phase angles resulting in the minimum and maximum values of Von Mises stress will satisfy the following relationship.

$$-\tan(2\theta) = \frac{|\sigma_{xx}|^2 \cdot \sin(2\delta) + |\sigma_{yy}|^2 \cdot \sin(2\beta) + 3 \cdot |\tau_{xy}|^2 \cdot \sin(2\phi)}{|\sigma_{xx}|^2 \cdot \cos(2\delta) + |\sigma_{yy}|^2 \cdot \cos(2\beta) + 3 \cdot |\tau_{xy}|^2 \cdot \cos(2\phi)} \; \frac{-|\sigma_{xx}|\,|\sigma_{yy}| \cdot \sin(\delta+\beta)}{-|\sigma_{xx}|\,|\sigma_{yy}| \cdot \cos(\delta+\beta)} \tag{8}$$

3.1 YIELDING CRITERION - DUCTILE MATERIAL (...)

With equation 8, it is possible to evaluate the reference angle (θ) for the calculation of the exact maximum equivalent Von Mises stress. From the maximum equivalent Von Mises stress, it is possible to calculate the exact minimum margin of safety from the following relation

$$M.S. = \frac{\sigma_{yield}}{\sigma_e} - 1 \tag{9}$$

Similar derivations have been performed for the 3D case with σ_{xx}, σ_{yy}, σ_{zz}, τ_{xy}, τ_{xz}, τ_{yz}.

3.2 INSTABILITY CRITERION - HONEYCOMB MATERIALS

In the case of local instability failure criteria such as in honeycomb materials, the fundamental relationships for the calculation of the margins of safety are directly evaluated from all stress field components. In the case of 2D element, the stress field is defined by the same relations as equations 2 thru 4. A typical and simplified relationship for the evaluation of the margin of safety associated with an instability failure in honeycomb materials is

$$\begin{gathered} M.S.(\theta) = \frac{1}{R_a(\theta) + R_s(\theta)} - 1.0 \\ R_a(\theta) = \frac{(\sigma_{xx}(\theta)^3 + \sigma_{yy}(\theta)^3)^{\frac{1}{3}}}{\sigma_a} \quad , \quad R_s(\theta) = \frac{\tau_{xy}(\theta)}{\tau_a} \end{gathered} \tag{10}$$

where

R_a	:	Compressive stress ratio
R_s	:	Shear stress ratio
σ_a	:	Compressive instability limit (pure mode)
τ_a	:	Shear instability limit (pure mode)

3.2 INSTABILITY CRITERION - HONEYCOMB MATERIAL (...)

The reference angle for the minimum margin of safety can be determined with the following relationship

$$\frac{d(M.S.)}{d(\theta)} = \frac{d}{d(\theta)}\left(\frac{1}{R_a(\theta)+R_s(\theta)}\right) = 0 \tag{11}$$

The solution of the above equation will lead to an expression for which it is impossible to get the critical values of the reference phase angle in an explicit form. In this case, equation 11 is numerically solved using a simple quadratic interpolation method for the calculation of the minimum margin of safety with the constraint that the stresses must be compressive since this is a requirement for an instability failure.

It is important to note that the above relationship is valid under the simplifying assumptions stated. In reality, the problem is much more complex [2] because the sign of σ_{xx} and σ_{yy} will introduce either a completely new relationship for the margin of safety calculation or correction terms to be added in equation 10.

3.3 OTHER CRITERIA

The above section presented the basic derivations for the calculation of the exact minimum margin of safety of a ductile material (Von Mises criterion) and honeycomb material (local instability failure). It is possible to extend the present method to brittle materials using typical failure theories such as the maximum principal stress theory, the Coulomb-Mohr theory or the modified Mohr theory.

4.0 MSC/NASTRAN POSTPROCESSOR

In order to implement the above theory, a MSC/NASTRAN post processor was developed. The software input and output files are shown in the next figure.

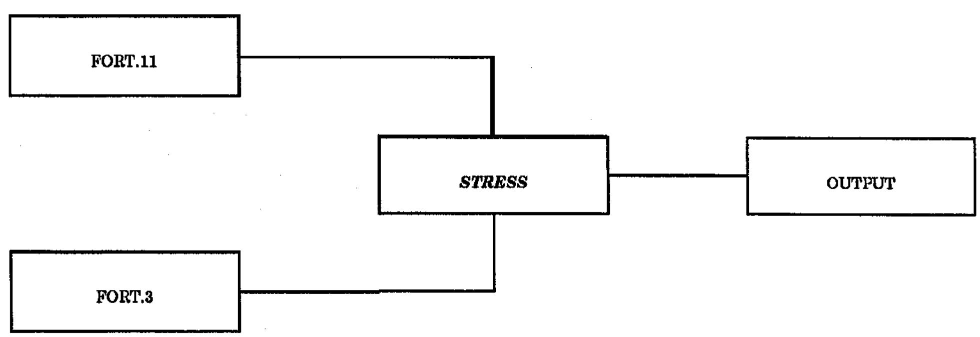


FIGURE 3 File structure for the MSC/NASTRAN post processor

4.0 MSC/NASTRAN POSTPROCESSOR (...)

The fort.11 is a binary file created by MSC/NASTRAN with the proper DMAP alter for static solutions (SOL 24, SOL 61 and SOL 101) or modal frequency response solutions (SOL30, SOL71 and SOL111). The binary files include the stress results requested by the ELSTRESS card in the case control deck. The elements supported by the software are :

1D ELEMENT	CROD,CBAR,CBEAM
2D ELEMENT	CTRIA3,CTRIA6,CQUAD4,CQUAD8
3D ELEMENT	CPENTA,CHEXA

The fort.3 is a user's input file which define the element ranges along with corresponding failures theories, range subtitles, factors of safety and allowable stresses.

The STRESS software is written in fortran 77. The processor is mainly used for the calculation of the minimum margin of safety for all the elements selected in the fort.3 file and to output the minumum margin of safety of all elements below a certain threshold.

5.0 RESULTS

The validation of the above expressions was performed by plotting either the maximum equivalent stress (ductile material case) or the minimum margin of safety as a function of ωt. From these plots, it was possible to evaluate the reference angle θ for which the maximum equivalent stress and consequently the minimum margin of safety occured. This process is illustrated in figure 4 using the example presented in section 2.2.

From equation 8, it is possible to calculate the reference phase angles associated with the stress field presented in section 2.2. The associated minimum and maximum equivalent Von Mises stress are then calculated from equation 5. The results obtained are as follows:

$\theta_1 = -4.75°$ $\qquad \sigma_{e1} = 8672.3$ PSI

$\theta_2 = 85.25°$ $\qquad \sigma_{e1} = 32206.1$ PSI

The comparison between these results with the minimum and the maximum values obtained from figure 4 show excellent agreement. It is possible to show the same agreement for all criteria implemented in the postprocessor.

5.0 RESULTS (...)

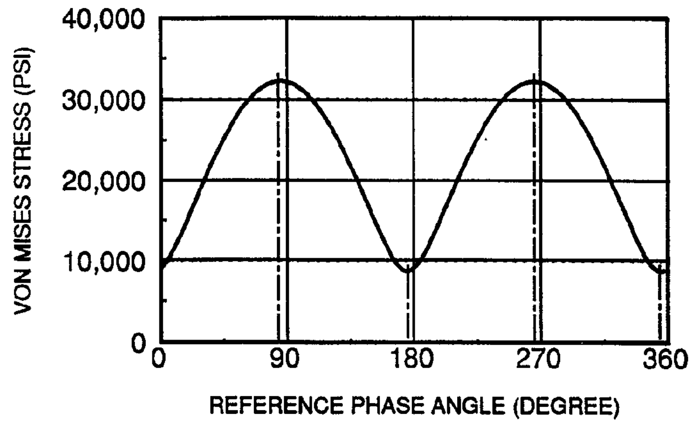


FIGURE 4 Equivalent Von Mises stress as a function of θ

6.0 DISCUSSIONS AND CONCLUSION

A general theory for the calculation of the minimum margin of safety for stress results obtained from a frequency response analysis was presented. In the case of ductile material (Von Mises criteria), an analytical relationship was derived. In the other cases, the minimum margin of safety is calculated with a numerical method based on simple quadratic interpolation.

The implementation of the theory was done through a postprocessor software. This method and the associated postprocessor software have been sucessfully implemented at Spar Aerospace Limited.

7.0 REFERENCES

[1] *MSC/NASTRAN User's Manual, Version 66*, The MacNeal-Schwendler corporation, Los Angeles, California, November 1988

[2] BRUHN, E.F., *Analysis and Design of Flight Vehicule Structures*, ed. Jacob Publishing Inc., 1973

[3] DIETER, George E., *Mechanical Metallurgy*, Ed. McGraw-Hill, Third edition, 1986, p.751.