Optimization of Damped Structures in the Frequency Domain

Tsung-hsiun Li and James E. Bernard

Department of Mechanical Engineering
Iowa State University
Ames, Iowa 50011, U.S.A.

ABSTRACT

This paper presents a way to efficiently compute the sensitivities of steady state resonant response and discusses the utility of these sensitivities in redesign and optimization. The resonant response sensitivities are calculated by combining the new capabilities of MSC/NASTRAN v67 in SOL 108, 111 and DMAP solution sequences. Two examples illustrate the approach.

Introduction

A large class of structures can be modelled as damped linear vibrational systems with harmonic excitation. It is often desirable to minimize the amplitudes of the response by modifying the design. Moore and Nagendra [1] introduced a general scheme to compute dynamic response sensitivities via a semi-analytical approach in both direct and modal formulations for frequency response. In the modal formulation, a technique was implemented which computed the response sensitivities without using the derivatives of eigenvectors. Watt and Starkey [2] presented a technique to modify the response amplitudes through changes in the mode shapes and damping, and explored the trade-offs between the cost of the design change and the size of improvement in the response. The methods of sensitivity analysis and optimization used in [1] and [2] are based on a fixed or constant driving frequency. Thomson and Bernard [3] computed derivatives of resonant response in the frequency domain. For derivatives of resonant response, the driving frequency will change as the design changes.

This paper presents a way to efficiently compute the derivatives of the amplitudes of the steady state resonant response of damped structures with respect to possible design changes. These sensitivities can be used to approximate resonant amplitudes to support the redesign process or to compute objective functions which depend on a resonant response. Two examples demonstrate the approach.

Sensitivity Derivatives of Resonant Response

Equations of Motion

The governing equations for frequency response of an N degrees of freedom linear vibratory system can be written:

$$(-\omega^2 [M] + j\omega [B] + [K])\{U\} = \{P\} \tag{1}$$

where $[M]_{n\times n}$, $[B]_{n\times n}$, and $[K]_{n\times n}$ are the mass, damping, and stiffness matrices, $\{U\}_{n\times 1}$ and $\{P\}_{n\times 1}$ are the vectors of response amplitudes and force magnitudes, and where ω is the driving frequency and $j = \sqrt{-1}$.

In MSC/NASTRAN, structural mass matrix $[M]$ is a sum of the contribution from element $[M_1]$ and the direct matrix input at grids $[M_2]$. The total damping force matrix $[B]$ is the sum of the contribution of viscous elements $[B_1]$ and the direct matrix input at grids $[B_2]$. The complete stiffness matrix $[K]$ for frequency response analysis consists of a superposition from the following sources:

$$[K] = [K_1] + jg[K_1] + \Sigma g_e k_e + [K_2] \tag{2}$$

where

$[K_1]$ is the structural stiffness matrix

g is the uniform structural damping coefficient, PARAM, G

g_e is the structural damping coefficient, MAT card.

$[K_2]$ is the direct matrix input at grids

The remainder of this section will discuss the sensitivity calculation of the steady state resonant response in both direct and modal formulations. The method employed here to calculate sensitivity derivatives is a semi-analytical approach including direct differentiation of the governing equations combined with difference approximations for first derivatives of the structural model.

Direct formulation

Let $\{e\}$ denote a vector of design parameters, $\{e^o\}$ be the initial design, and ω^o be a natural frequency of the initial design. For resonance, the mass, damping, stiffness matrices and the driving frequency ω_i are functions of design variables e. Equation (1) can be written as:

$$[Z(\omega_i(e), e)]\{U(e)\} = \{P(e)\} \tag{3}$$

where

$$[Z(\omega_i(e), e)] = -\omega^2{}_i(e)[M(e)] + j\omega_i(e)[B(e)] + [K(e)] \tag{4}$$

Taking partial derivatives of Equation (3) with respect to design variables e_k, where k=1,2...s, we get

$$[Z]\frac{\partial}{\partial e_k}\{U\} = \frac{\partial}{\partial e_k}\{P\} - \left(\frac{\partial}{\partial e_k}[Z]\bigg|_{\omega_i = \omega_i^o} + \frac{\partial}{\partial \omega_i}[Z] \cdot \frac{\partial \omega_i}{\partial e_k}\bigg|_{e = e^o}\right)\{U\} \tag{5}$$

Define

$$[Z]\frac{\partial}{\partial e_k}\{U\} = [Z]\frac{\partial}{\partial e_k}\{U\}^C + [Z]\frac{\partial}{\partial e_k}\{U\}^V \tag{6}$$

where $\frac{\partial}{\partial e_k}\{U\}^C$ is the sensitivity of resonant response with a constant driving frequency and $\frac{\partial}{\partial e_k}\{U\}^V$ represents the sensitivity of resonant response due to the change in resonant frequency, i.e.,

$$[Z]\frac{\partial}{\partial e_k}\{U\}^C = \frac{\partial}{\partial e_k}\{P\} - \left(\frac{\partial}{\partial e_k}[Z]\bigg|_{\omega = \omega_i^o}\right)\{U\} \tag{7a}$$

$$[Z]\frac{\partial}{\partial e_k}\{U\}^V = -\left(\frac{\partial}{\partial \omega_i}[Z] \cdot \frac{\partial \omega_i}{\partial e_k}\bigg|_{e = e^o}\right)\{U\} \tag{7b}$$

where

$$\frac{\partial}{\partial e_k}[Z]\bigg|_{\omega = \omega_i^o} = -\omega_i^2\frac{\partial}{\partial e_k}[M] + j\omega_i\frac{\partial}{\partial e_k}[B] + \frac{\partial}{\partial e_k}[K] \tag{8}$$

$$\frac{\partial}{\partial \omega_i}[Z] \cdot \frac{d\omega_i}{de_k}\bigg|_{e = e^o} = (-2\omega_i[M] + j[B])\frac{\partial \omega_i}{\partial e_k} \tag{9}$$

For small damping, we do not expect the resonant frequency to be damping dependent. For an undamped vibratory system, it is well known [4] that

$$\frac{\partial \omega^2_i}{\partial e_k} = \frac{\Phi_i^t(\frac{\partial}{\partial e_k}[K] - \omega_i^2\frac{\partial}{\partial e_k}[M])\Phi_i}{\Phi_i^t[M]\Phi_i} \tag{10}$$

where Φ_i are the eigenvectors of the system. Thus,

$$\frac{\partial \omega_i}{\partial e_k} = \left(\frac{\Phi_i^t(\frac{\partial}{\partial e_k}[K] - \omega_i^2\frac{\partial}{\partial e_k}[M])\Phi_i}{\Phi_i^t[M]\Phi_i}\right) \cdot \left(\frac{1}{2\omega_i}\right) \tag{11}$$

Note that right hand side of the calculation called for by Equation (5) includes only products of vectors and matrices, i.e., they take on the order of N^2 calculations. The solution for $\frac{\partial}{\partial e_k}\{U\}$ can then also be done in $O(N^2)$ calculations because the decomposition of $[Z]$ is known from the solution of Equation (3).

Modal Formulation

For large-scale dynamic systems, modal reduction is a commonly used technique to reduce computational effort. Assume that

$$\{U\}_{n\times 1} = [\Phi]_{n\times m}\cdot\{\xi\}_{m\times 1} \tag{12}$$

and

$$\frac{\partial}{\partial e_k}\{U\}_{n\times 1} = [\Phi]_{n\times m}\cdot\{q_k\}_{m\times 1} \tag{13}$$

where Φ is the reduced modal matrix from real eigenvalue analysis $([K]-\omega_i^2[M])\Phi_i = 0$ $i=1,\ldots,m$, and $\{\xi\}$ represents the reduced response vector from modal frequency response analysis. By substituting Equations (12) and (13) into Equation (5) and pre-multiplying both sides by Φ^t, we have

$$[\tilde{Z}]\{q_k\} = \Phi^t\frac{\partial}{\partial e_k}\{P\} - \Phi^t\left(\frac{\partial}{\partial e_k}[Z]\bigg|_{\omega_i=\omega_i^o} + \frac{\partial}{\partial\omega_i}[Z]\cdot\frac{\partial\omega_i}{\partial e_k}\bigg|_{e=e^o}\right)(\Phi\cdot\{\xi\}) \tag{14}$$

where

$$[\tilde{Z}]_{m\times m} = -\omega_i^2(\Phi^t[M]\Phi) + j\omega_i(\Phi^t[B]\Phi) + (\Phi^t[K]\Phi) \tag{15}$$

Define

$$[\tilde{Z}]\{q_k\} = [\tilde{Z}](\{q_k\}^C + \{q_k\}^V) \tag{16}$$

where

$$[\tilde{Z}]\{q_k\}^C = \Phi^t\frac{\partial}{\partial e_k}\{P\} - \Phi^t\left(\frac{\partial}{\partial e_k}[Z]\bigg|_{\omega_i=\omega_i^o}\right)(\Phi\cdot\{\xi\}) \tag{17}$$

$$[\tilde{Z}]\ \{q_k\}^V = -\left(\Phi^t\left(\frac{\partial}{\partial\omega_i}[Z]\cdot\left.\frac{\partial\omega_i}{\partial e_k}\right|_{e=e^o}\right)\Phi\right)\{\xi\} \qquad (18)$$

Equations (17) and (18) can be rewritten as

$$[\tilde{Z}]\ \{q_k\}^C = \Phi^t\frac{\partial}{\partial e_k}\{P\} - \Phi^t\left(-\omega_i^2\frac{\partial}{\partial e_k}[M] + j\omega_i\frac{\partial}{\partial e_k}[B] + \frac{\partial}{\partial e_k}[K]\right)(\Phi\cdot\{\xi\}) \qquad (19)$$

$$[\tilde{Z}]\ \{q_k\}^V = -\left((-2\omega_i(\Phi^t[M]\Phi) + j(\Phi^t[B]\Phi))\frac{d\omega_i}{de_k}\right)\{\xi\} \qquad (20)$$

Therefore, the sensitivities of resonant response can be obtained from the following relationship.

$$\frac{\partial}{\partial e_k}\{U\} = \frac{\partial}{\partial e_k}\{U\}^C + \frac{\partial}{\partial e_k}\{U\}^V = \Phi\cdot(\{q_k\}^C + \{q_k\}^V) \qquad (21)$$

Optimization in the Frequency Domain

Finite-element based optimization usually entails several challenges including formulation of the objective function, design sensitivity calculations, effectively using these sensitivities to approximate finite element results called for by the objective function, minimization of the objective function to determine the optimal design, and checking the optimal design to see if the sensitivities led to an accurate solution. This paper will demonstrate the utility of resonant response sensitivities to replace recalculating resonant response for each new design.

Typically, Equation (1) yields complex amplitudes. Assume that u_i is the i^{th} entry of the vector of response amplitudes $\{U\}$ and y_i is the magnitude of u_i, then a simple relationship between $\frac{dy_i}{de_k}$ and $\frac{du_i}{de_k}$ can be derived:

$$y_i = \sqrt{(Re(u_i))^2 + (Im(u_i))^2} \qquad (22)$$

$$\frac{\partial y_i}{\partial e_k} = \left[Re(u_i)Re\left(\frac{\partial u_i}{\partial e_k}\right) + Im(u_i)Im\left(\frac{\partial u_i}{\partial e_k}\right)\right]\cdot(1/y_i) \qquad (23)$$

Now the i^{th} amplitude of resonant response can be approximated by

$$y_i = y^o{}_i + \sum_1^s\frac{\partial y_i}{\partial e_k}\cdot(e_k - e^o{}_k) \qquad (24)$$

Numerical Examples

Two examples will illustrate the meaning of the resonant response sensitivities and the application of these sensitivities in redesign process and optimization procedure. Example 1 is a simple two degree-of-freedom spring-mass vibratory system. This simple example has a closed-form solution which allows analytical verification of the finite-element-based sensitivity calculations of resonant response. The second is a clamped plate. This example will show that the method developed here can be applied to a damped large degree-of -freedom system.

Example 1: A Two Degree-of-Freedom Spring-Mass System

Figure 1 presents the configuration of a simple two degrees of freedom system which has a harmonic loading F_1 on mass 1, where $F_1 = f_1 \cdot e^{j\omega_1 t}$. The governing equations for the steady state frequency response of this system are

$$\left(-\omega_1^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + j\omega_1\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\right) \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \end{bmatrix} \tag{25}$$

where $\{X\}$ represents the response amplitudes of the displacement vector $\{x\}$, and where the driving frequency is at the first natural frequency ω_1 and $j = \sqrt{-1}$. With the given values, this yields for excitation at first mode resonance

$$\left(-\omega_1^2\begin{bmatrix} 3.0 & 0 \\ 0 & 2.0 \end{bmatrix} + j\omega_1\begin{bmatrix} 0.15 & -0.05 \\ -0.05 & 0.15 \end{bmatrix} + \begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix}\right) \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0 \end{bmatrix} \tag{26}$$

where $\omega_1 = 2\pi f_1$ and $f_1 = f_1^o = 0.14$ (hz).

First, assume that m_1 is the only design variable. Figure 2 presents the topology of resonant response amplitude $|X_1|$ across a design range of m_1 from 2.0 to 4.0 and a driving frequency range of f_d from 0.0 to 0.4 (hz). This figure clearly shows the relationship between m_1 and the two resonant frequencies. The original resonant amplitude X_1 = 8.27E-02 - 7.50E+00 j with $|X_1^o|$ = 7.50. Using Equation (6) to compute the sensitivity of resonant response amplitude $|X_1|$ yields

$$\frac{dX_1}{dm_1} = \frac{dX_1^C}{dm_1} + \frac{dX_1^V}{dm_1} = (\text{-}4.34\text{E}{+}01 - 9.56\text{E-}01j) + (4.33\text{E}{+}01 - 9.47\text{E-}01j) = \text{-}1.0\text{E-}1 - 1.90\text{E}{+}00\,j \qquad (27)$$

Equation(23) yields the sensitivity of the magnitude of the resonant response.

$$\frac{d}{de_k}|X_1| = \left[Re\,(X_1)\,Re\left(\frac{dX_1}{dm_1}\right) + Im\,(X_1)\,Im\left(\frac{dX_1}{dm_1}\right)\right] \cdot (1/|X_1|) = 1.90 \qquad (28)$$

Two observations follow from these numerical results. First note that both X_1 and its derivative are dominated by the imaginary part. This is to be expected for resonant response. Secondly note that $\frac{dX_1^C}{dm_1}$, the sensitivity at constant driving frequency, is not useful by itself - the varying frequency term is an important part of the amplitude sensitivity. Figure 3 presents resonant values of $|X_1|$ at various values of m_1. The dashed line indicates that the resonant amplitude $|X_1|$ calculated using the resonant response sensitivity gives a very good prediction for a design change within ten percent of the original m_1. Figure 4 compares a prediction of resonant response amplitude $|X_1|$ with exact results across a wider range of m_1. The figure indicates that resonant response sensitivity gives a reasonable approximation even across this large range.

Now consider a more complicated example in which all masses and springs are potential design changes. Table 1 compares the sensitivities of resonant response amplitude $|X_1|$ calculated from closed-form solution to the MSC/NASTRAN-based results. By a procedure related to steepest decent, we select the step size of design change based on resonant response sensitivities normalized to the largest sensitivity, $\frac{\partial}{\partial k_1}|X_1|$. Table 2 presents the computed step size for each design variable and the predicted resonant response amplitude $|X_1|$ based on 10% and 50% perturbed step sizes of k_1 and Figure 5 compares exact and predicted amplitudes of resonant response. The table and figure indicate that the sensitivity was a good approximation for 10% change. Furthermore, although the approximation seems to have broken down for the 50% change, the resonant response amplitude based on the overall steepest decent direction was driven down by a factor about 3.

Consider optimization with the goal of minimizing the first mode resonant response of both X_1 and X_2. Following reference [5], we choose the simple objective function

$$C = \sum_{1}^{5} a_j \cdot (e_j - e_j^o)^2 + \sum_{1}^{2} b_i \cdot \left(|X_i^o| + \sum_{1}^{5} \frac{\partial}{\partial e_k} |X_i| \cdot (e_k - e^o{}_k) \right) \tag{29}$$

Table 3 presents the improved design and Figures 6 and 7 present the improved frequency response. The changes are large, thus it is not surprising that the approximation, as indicated by the box in Figure 6, is not directly an target. Nevertheless, the overall results, as indicated by the exact response of both amplitudes are clearly a step in the right direction.

Example 2: A Clamped Plate Subject To A Harmonic Loading

A steel plate of thickness 0.25 in., width 20 in, and length 90 in., clamped along two edges is used to support a 60 lbs machine which is mounted in the middle of the plate. See Figure 8. During the operation of this machine, the plate is subjected to a harmonic load, $P(t) = 20.0e^{j\omega t}$ lbs, with driving frequency ω at the first natural frequency ω_1 of the initial system. This plate is divided into nine sections and the thickness of each section is selected a design variable. The plate is modelled as a finite element mesh with 288 CQUAD4 elements with 1575 degrees of freedom. This model also includes a single scalar mass element at the center with a weight 60 lbs. We assume the uniform structural damping equivalent to damping ratio 0.025 (PARAM,G 0.05). Figure 9 shows the finite element mesh of the plate and its first mode shape. This example illustrates the application of a modal reduction scheme in computing the resonant response sensitivities with driving frequency tracking. The reduced modal matrix Φ of this system was formed by the first 20 eigenvectors.

Table 4 presents the sensitivities of the amplitude of resonant response at the center of the plate, $|U_z 167|$. In the redesign procedure, we use these sensitivities to scale the design change based on t_5. Figure 10 and Table 5 present predicted and exact response amplitudes $|U_z 167|$ with 10% and 50% changes of t_5. The inaccuracy of the linear result for the 50% design change indicated that, as we might expect, the amplitude of resonant response is a nonlinear function of the thicknesses.

Conclusion and Ongoing Research

This paper presented a MSC/NASTRAN-based technique to compute the first order sensitivity of the steady state resonant response with respect to design variables. Two examples illustrate that such sensitivities may be useful across fairly large design changes.

Our ongoing work is concerned with large changes: To get improved accuracy, there is a choice between using higher order terms and intermittent recalculation of the frequency response and linear sensitivities. The ultimate objective is to blend these sensitivities into a graphics-based interface which will facilitate better participation of the designer in the optimization process.

References

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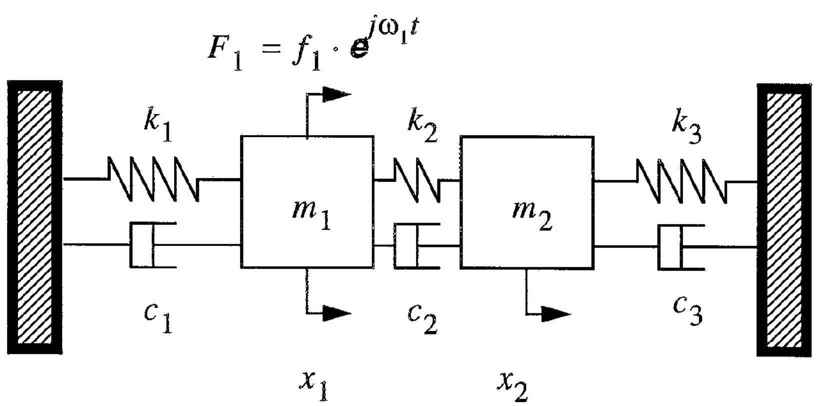
[7] MSC/NASTRAN User's Manual vol. I & II, Version 67, The MacNeal Schwendler Corporation, Los Angels, CA, August 1991.

[8] MSC/NASTRAN Application Manual vol. II, Version 67, The MacNeal Schwendler Corporation, Los Angels, CA, October 1991.

[9] IMSL MATH/LIBRARY User's Manual vol. 3, chapter 8.

Figure-1. A two degree-of-freedom spring-mass system.



Assume $\{x\} = \{X\} \cdot e^{j\omega_1 t}$

Design Vector $\{e\} = [m_1, m_2, k_1, k_2, k_3]$

Initial Design $\{e^o\} = [3.0, 2.0, 2.0, 1.0, 2.0]$

and $[c_1, c_2, c_3] = [0.1, 0.05, 0.1]$

Figure-2. The Topology of amplitude of resonant response $|X_1|$

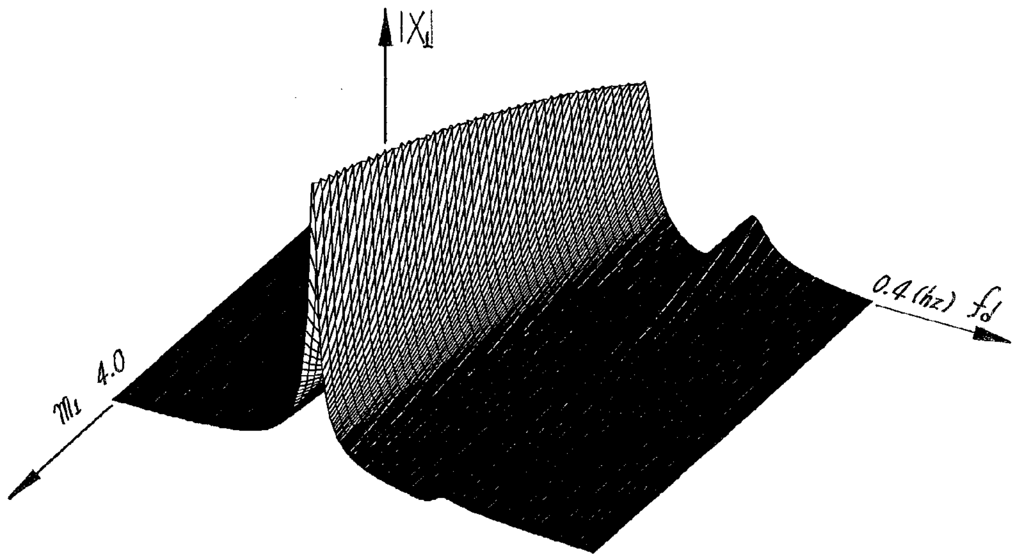


Figure-3. Amplitude of resonant response $|X_1|$ with various values of m_1

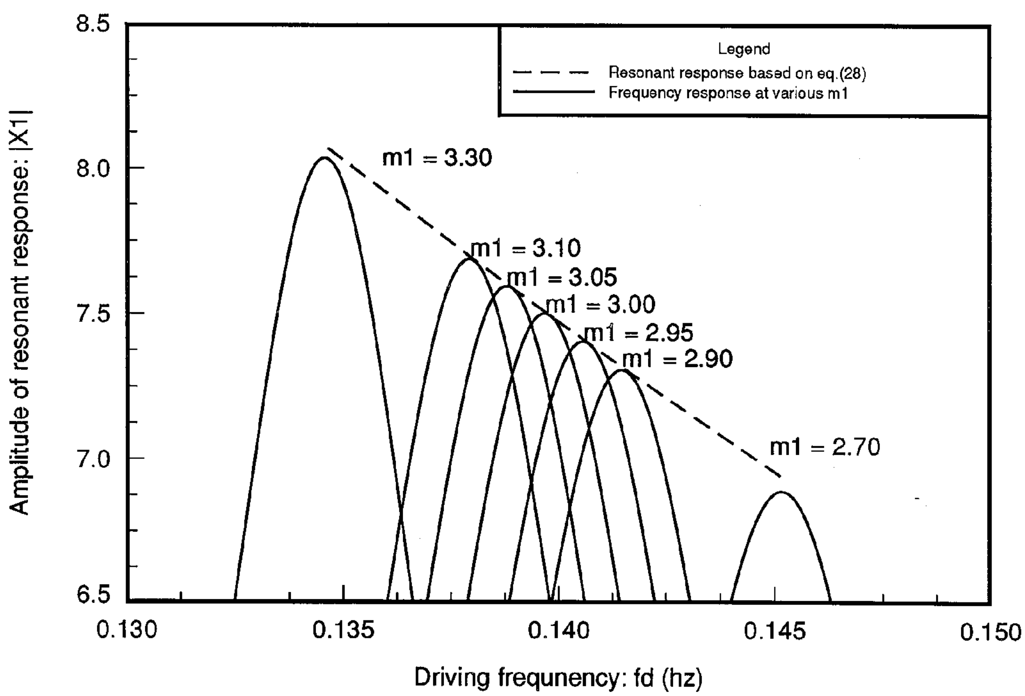


Figure-4. A comparison of predicted and exact amplitudes of resonant response $|X_1|$ across a range of m_1

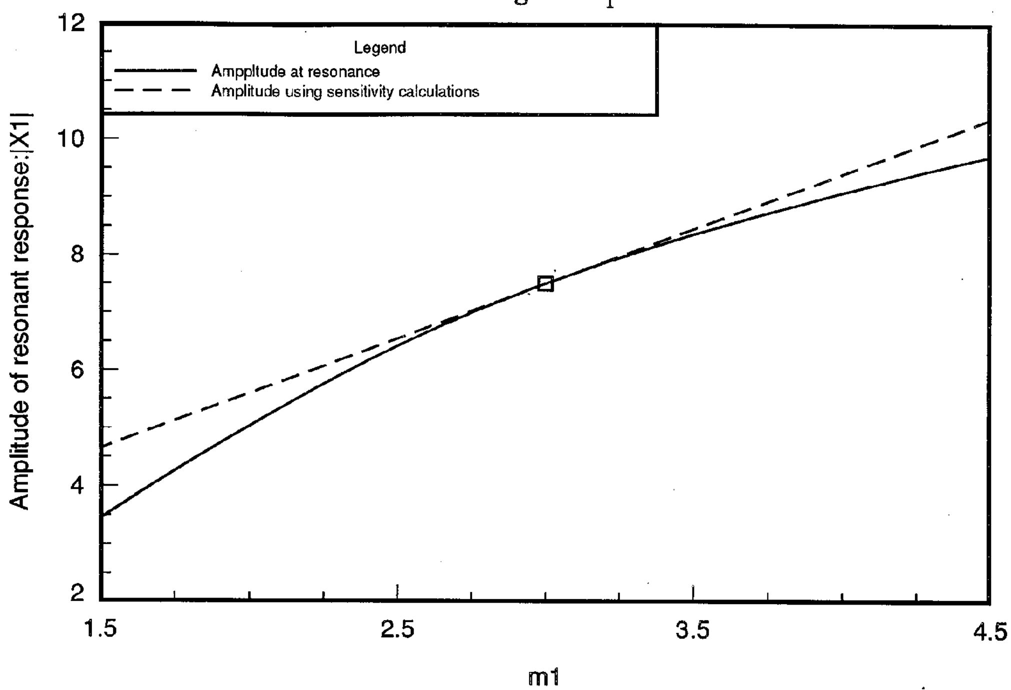


Figure-5. Predicted and exact amplitudes of resonant response $|X_1|$ with various step sizes of design changes

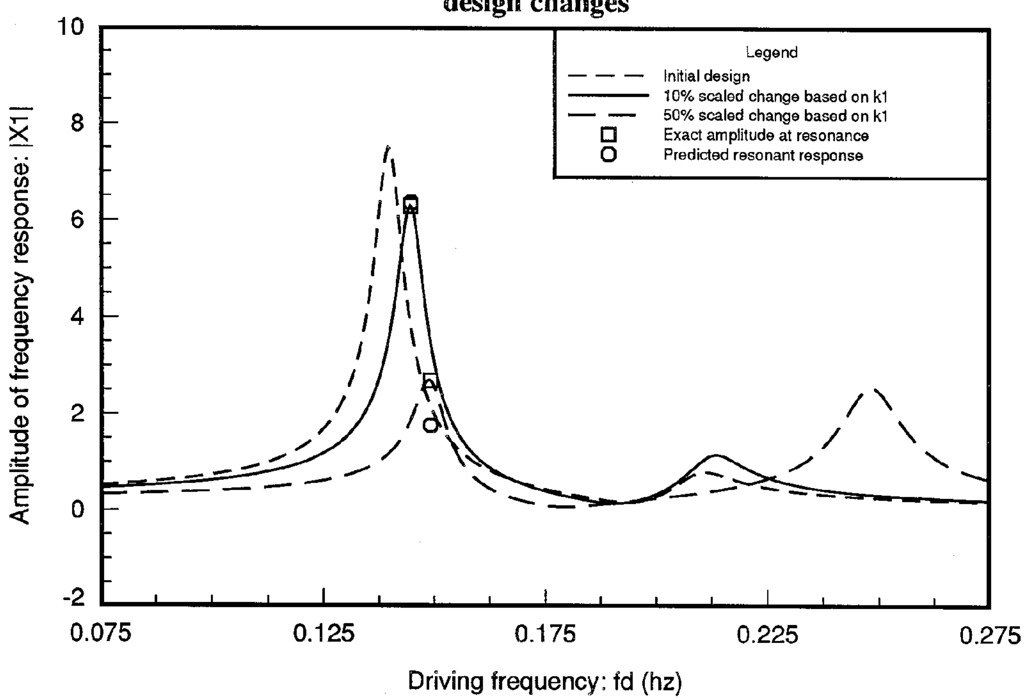


Table-1 Sensitivities of the amplitude of resonant response $|X_1|$

e_k	$\frac{\partial}{\partial e_k}\lvert X_1^C\rvert$ EXACT SOLUTION	$\frac{\partial}{\partial e_k}\lvert X_1^C\rvert$ MSC/ NASTRAN	$\frac{\partial}{\partial e_k}\lvert X_1\rvert$ EXACT SOLUTION	$\frac{\partial}{\partial e_k}\lvert X_1\rvert$ MSC/ NASTRAN
m_1	0.4782	0.4780	1.9016	1.9014
m_2	-1.6514	-1.6515	-0.9813	-0.9814
k_1	-0.6199	-0.6197	-2.4654	-2.4653
k_2	-1.1882	-1.1879	-1.3700	-1.3697
k_3	2.1411	2.1413	1.2723	1.2725

Table-2 Using scaled step sizes of design change to compare the predicted and exact amplitudes of resonant response $|X_1|$

Items	Initial values	Change normalized to a 10% change of k_1	Change normalized to a 50% change of k_1
m_1	3.0000	2.8457	2.2287
m_2	2.0000	2.0796	2.3981
k_1	2.0000	2.2000	3.0000
k_2	1.0000	1.1111	1.5556
k_3	2.0000	1.8968	1.4838
f_1 (hz)	0.139775	0.144720	0.149203
$\lvert X_1\rvert_{predicted}$	---------------	6.349884	1.757530
$\lvert X_1\rvert_{exact}$	7.497949	6.287626	2.686227
$\Delta\lvert X_1\rvert$	---------------	0.0623	-0.9287
Error%	---------------	0.9902	-34.5725

Figure-6. Optimization result of the two degree-of-freedom spring-mass System

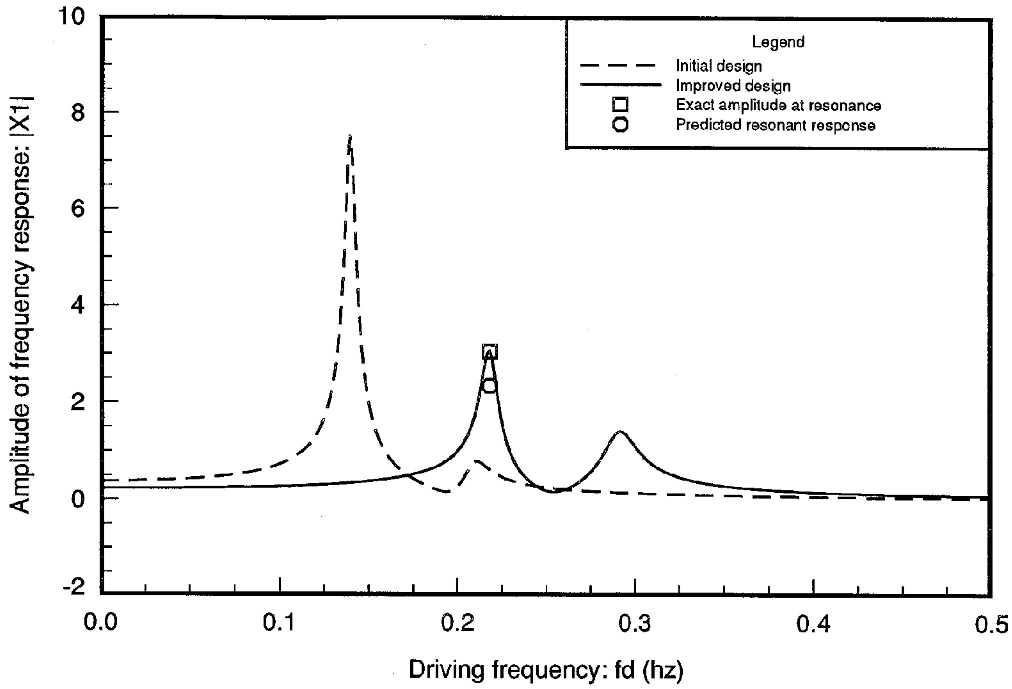


Figure-7. Optimization result of the two degree-of-freedom spring-mass System

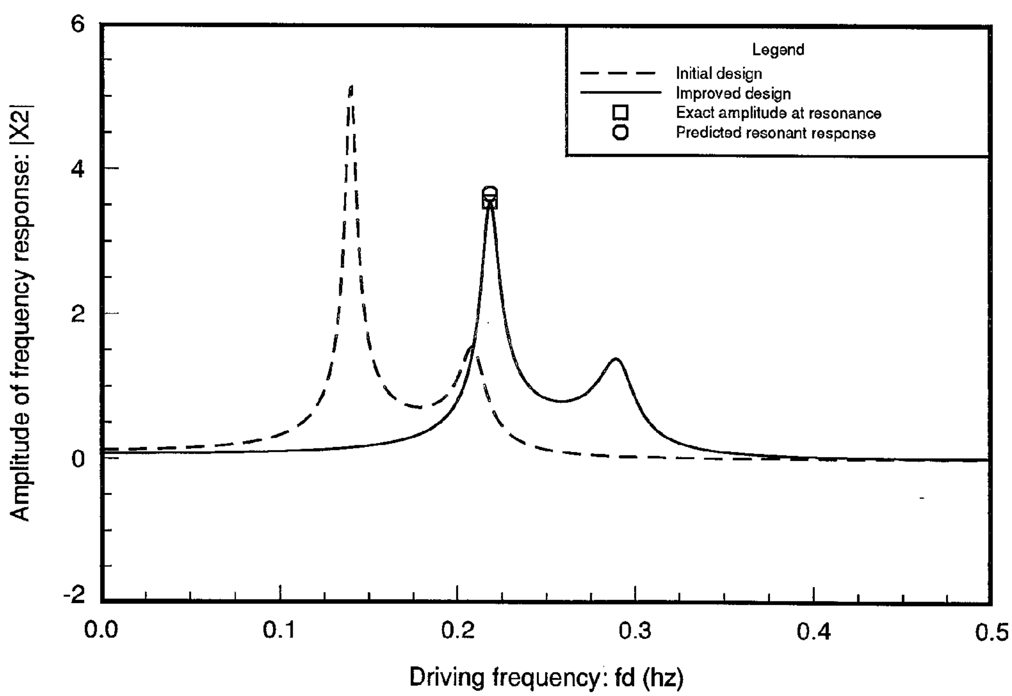


Table-3. Optimization result of the two degree-of-freedom spring-mass System

	Weighting parameters a_i, b_j	Initial design	Improved design
m_1	0.7500	3.00000	1.78355
m_2	0.7500	2.00000	1.50454
k_1	0.7500	2.00000	3.57716
k_2	0.7500	1.00000	1.19194
k_3	0.7500	2.00000	2.64234
$\|X_1\|_{predicted}$	1.0000	----------	2.33746
$\|X_1\|_{exact}$	----------	7.49795	3.04915
$\|X_2\|_{predicted}$	1.2500	----------	3.66256
$\|X_2\|_{exact}$	----------	5.12870	3.55067

Figure-8. A clamped plate subject to a harmonic loading P

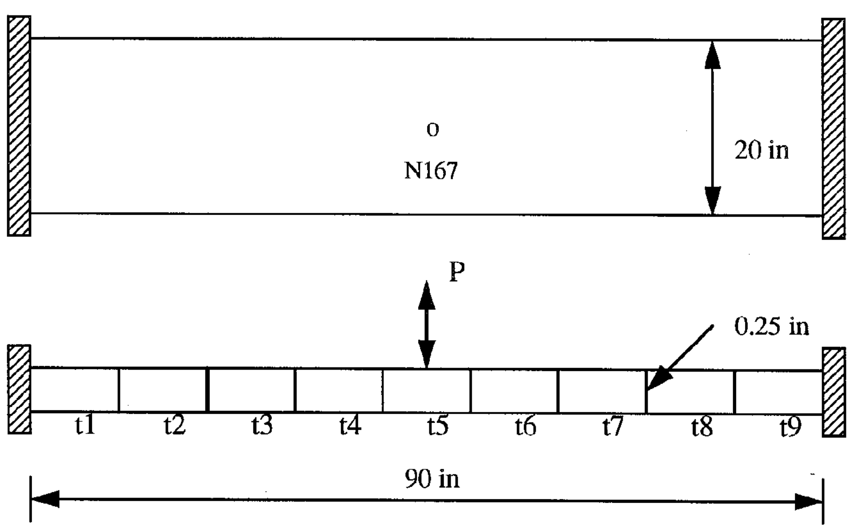


Figure-9. Finite element model of the plate and its first mode shape

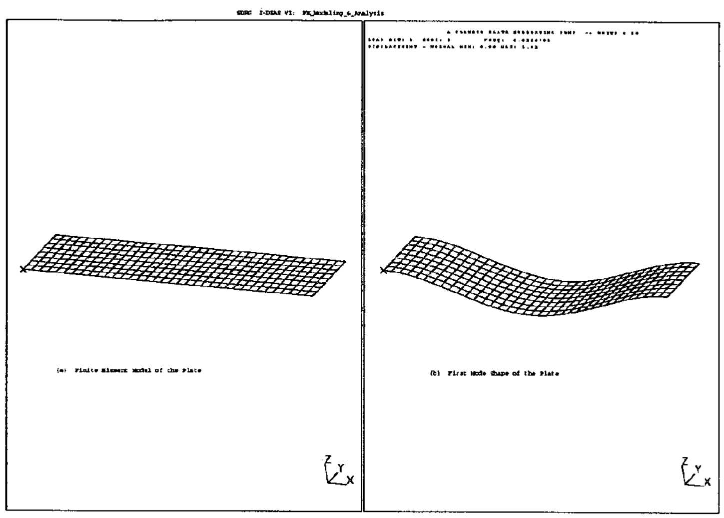


Figure-10. Predicted and exact amplitudes of resonant response $|U_z 167|$ with various step sizes of design changes

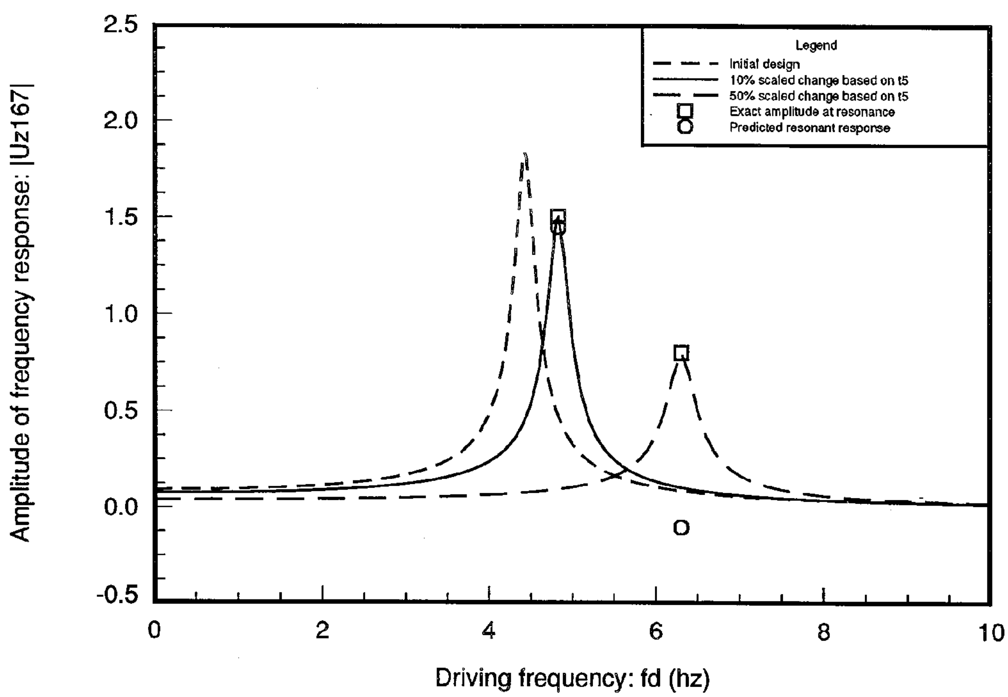


Table-4. Sensitivities of the amplitude of resonant response $|U_Z 167|$: $\frac{\partial}{\partial e_k}|U_z 167|$

e_k	MSC/ NASTRAN	FINITE DIFF.
t_1	-4.6062	-4.5600
t_2	-0.9962	-1.0000
t_3	-0.2380	-0.2359
t4	-2.4078	-2.3850
t_5	-5.7821	-5.7600
t_6	-2.4078	-2.3850
t_7	-0.2380	-0.2359
t_8	-0.9962	-1.0000
t_9	-4.6062	-4.5600

Table-5. Using scaled step sizes of design change to compare the predicted and exact amplitudes of resonant response $|U_Z 167|$

e_k	Initial design	Change normalized to a 10% change of t_5	Change normalized to a 50% change of t_5
t_1	0.250000	0.269916	0.349579
t_2	0.250000	0.254307	0.271536
t_3	0.250000	0.251029	0.255145
t_4	0.250000	0.260411	0.302053
t_5	0.250000	0.275000	0.375000
t_6	0.250000	0.260411	0.302053
t_7	0.250000	0.251029	0.255145
t_8	0.250000	0.254307	0.271536
t_9	0.250000	0.269916	0.349579
f_1 (hz)	4.429474	4.828553	6.305948
$\|U_z 167\|_{predicted}$	----------	1.447824	-0.101091
$\|U_z 167\|_{exact}$	1.835057	1.505232	0.804276