AN ALTERNATIVE APPROACH FOR IMPROVING THE ACCURACY OF SEMI-ANALYTIC DESIGN SENSITIVITIES IN SHAPE OPTIMIZATION

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ABSTRACT

The semi-analytic method has been adopted for shape sensitivity calculations because it does not require special code for analytic derivatives of element matrices. However, this method may have serious accuracy problems for which several approaches have been proposed. These approaches, however, are either lack of generality or difficult to be integrated with a general-purpose FEM package.

Here, an effective approach to the problem has been developed in which an iterative algorithm is used. It not only improves the accuracy but also provides error estimators so as to ensure the quality of calculated sensitivities. In addition, it can be easily integrated with MSC/NASTRAN. In this paper, the basic idea of the approach is first described. Then, a general algorithm based on the approach is given. Finally, its effectiveness is shown through numerical results.

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1. INTRODUCTION

The semi-analytic method has been adopted in MSC/NASTRAN for shape sensitivity calculation because it does not require analytic derivatives of element matrices which depend on element shape functions. Since it is not easy or not possible to obtain analytic expressions for some advanced elements, all the elements are processed with semi-analytic method in design sensitivity analysis [1]. However, this method may have serious accuracy problems [2]. References 2 through 8 have proposed several methods to address the problems. These approaches, however, are either lack of generality or difficult to be integrated into MSC/NASTRAN. Recently, the authors have presented an iterative approach used with the forward difference scheme for semi-analytic sensitivities [9]. It should be pointed out that the similar idea has also been described in Ref.2.

Compared with the forward difference approximation, central differencing produces one order higher accurate results. It has been used successfully for semi-analytic sensitivities of a car model as described in Ref.3. However, the accuracy of results is unknown and no error estimator is provided by the central difference approach.

This paper extends the iterative approach for semi-analytic sensitivities to the central difference scheme. The modified central difference approach not only improves the accuracy but also provides error estimators so as to ensure the quality of calculated sensitivities. In this paper, the basic idea of the approach is first described. Then, a general algorithm based on the approach is given. Finally, its effectiveness is shown through numerical results.

2. MODIFIED CENTRAL DIFFERENCE APPROACH

2.1 Traditional Central Difference Approach

When the central difference approximation is used, the semi-analytic displacement sensitivities are calculated by the equation:

$$\mathbf{K}\frac{\Delta \mathbf{U}}{\Delta \mathrm{S}} = \frac{\Delta \mathbf{P}}{\Delta \mathrm{S}} - \frac{\Delta \mathbf{K}}{\Delta \mathrm{S}}\mathbf{U} \tag{1}$$

where

$$\frac{\Delta \mathbf{K}}{\Delta \mathrm{S}} = \frac{\mathbf{K}(\mathrm{S}+\Delta \mathrm{S}) - \mathbf{K}(\mathrm{S}-\Delta \mathrm{S})}{2\Delta \mathrm{S}} \tag{2}$$

where $\mathbf{K}$, $\mathbf{U}$ and $\mathbf{P}$ are stiffness matrix, displacement vector and applied force vector of the system of interest, S a design variable. These prefixed with Δ are their perturbed terms corresponding to a small perturbation of the design variable ΔS.

Although Eqs.(1) and (2) can be used to improve the accuracy of sensitivities, results are not ensured because an ideal step size ΔS is difficult to estimate in advance. Further more, no error measure is provided for estimating any sensitivity error.

Following sections will present a modified central difference approach which will not only improve the accuracy of sensitivities but also provide error estimators.

2.2 Modified Central Difference Approach

For two small perturbations about a known solution, the matrix equations of equilibrium of system can be expressed as:

$$\mathbf{K}^{-}\mathbf{U}^{-} = \mathbf{P}^{-} \tag{3}$$

$$\mathbf{K}^{+}\mathbf{U}^{+} = \mathbf{P}^{+} \tag{4}$$

An 'exact result' would be the difference of the two solutions, $\mathbf{U}^{+} - \mathbf{U}^{-}$, which requires expensive decomposition of $\mathbf{K}^{+}$ and $\mathbf{K}^{-}$ and should be avoided. Herein, we will expand Eqs.(3) and (4) to obtain finite difference result as follows.

Using Taylor's theorem, one can expand $\mathbf{K}^{-}$ as

$$\begin{aligned}
\mathbf{K}^{-} &= \mathbf{K}(\mathrm{S}-\Delta \mathrm{S}) \\
&= \mathbf{K} - \Delta \mathrm{S}\frac{\partial \mathbf{K}}{\partial \mathrm{S}} + \frac{\Delta \mathrm{S}^2}{2}\frac{\partial^2 \mathbf{K}}{\partial \mathrm{S}^2} - \frac{\Delta \mathrm{S}^3}{6}\frac{\partial^3 \mathbf{K}}{\partial \mathrm{S}^3} + \frac{\Delta \mathrm{S}^4}{24}\frac{\partial^4 \mathbf{K}}{\partial \mathrm{S}^4} \\
&\quad + \cdots + (-1)^k \frac{\Delta \mathrm{S}^k}{k!}\frac{\partial^k \mathbf{K}}{\partial \mathrm{S}^k} + \cdots
\end{aligned}$$

$$= \mathbf{K} - \Delta\mathbf{K}^o + \Delta\mathbf{K}^e \tag{5}$$

where $\Delta\mathbf{K}^o$ is the sum of odd-numbered perturbed terms given by

$$\Delta\mathbf{K}^o = \Delta S\frac{\partial\mathbf{K}}{\partial S} + \frac{\Delta S^3}{6}\frac{\partial^3\mathbf{K}}{\partial S^3} + \cdots + \frac{\Delta S^{(2k-1)}}{(2k-1)!}\frac{\partial^{(2k-1)}\mathbf{K}}{\partial S^{(2k-1)}} + \cdots, \quad k = 1, 2, 3, \cdots \tag{6}$$

and $\Delta\mathbf{K}^e$ is the sum of even-numbered perturbed terms as

$$\Delta\mathbf{K}^e = \frac{\Delta S^2}{2}\frac{\partial^2\mathbf{K}}{\partial S^2} + \frac{\Delta S^4}{24}\frac{\partial^4\mathbf{K}}{\partial S^4} + \cdots + \frac{\Delta S^{(2k)}}{(2k)!}\frac{\partial^{(2k)}\mathbf{K}}{\partial S^{(2k)}} + \cdots, \quad k = 1, 2, 3, \cdots \tag{7}$$

Similarly, $\mathbf{K}^+$, $\mathbf{U}^-$, $\mathbf{U}^+$, $\mathbf{P}^-$ and $\mathbf{P}^+$ can be written as

$$\mathbf{K}^+ = \mathbf{K} + \Delta\mathbf{K}^o + \Delta\mathbf{K}^e \tag{8}$$

$$\mathbf{U}^- = \mathbf{U} - \Delta\mathbf{U}^o + \Delta\mathbf{U}^e \tag{9}$$

$$\mathbf{U}^+ = \mathbf{U} + \Delta\mathbf{U}^o + \Delta\mathbf{U}^e \tag{10}$$

$$\mathbf{P}^- = \mathbf{P} - \Delta\mathbf{P}^o + \Delta\mathbf{P}^e \tag{11}$$

$$\mathbf{P}^+ = \mathbf{P} + \Delta\mathbf{P}^o + \Delta\mathbf{P}^e \tag{12}$$

where $\Delta\mathbf{U}^o, \Delta\mathbf{U}^e, \Delta\mathbf{P}^o$ and $\Delta\mathbf{P}^e$ are obtained from Eqs.(6) and (7) by replacing $\mathbf{K}$ with $\mathbf{U}$ and $\mathbf{P}$, respectively. Substituting Eqs.(5) and (8 $\sim$ 12) into Eqs.(3) and (4), one gets

$$(\mathbf{K} - \Delta\mathbf{K}^o + \Delta\mathbf{K}^e)(\mathbf{U} - \Delta\mathbf{U}^o + \Delta\mathbf{U}^e) = \mathbf{P} - \Delta\mathbf{P}^o + \Delta\mathbf{P}^e \tag{13}$$

$$(\mathbf{K} + \Delta\mathbf{K}^o + \Delta\mathbf{K}^e)(\mathbf{U} + \Delta\mathbf{U}^o + \Delta\mathbf{U}^e) = \mathbf{P} + \Delta\mathbf{P}^o + \Delta\mathbf{P}^e \tag{14}$$

Expanding these two equations, one can have

$$\mathbf{K}\Delta\mathbf{U}^o + \Delta\mathbf{K}^o\mathbf{U} + \Delta\mathbf{K}^o\Delta\mathbf{U}^e + \Delta\mathbf{K}^e\Delta\mathbf{U}^o = \Delta\mathbf{P}^o \tag{15}$$

$$\mathbf{K}\Delta\mathbf{U}^e + \Delta\mathbf{K}^e\mathbf{U} + \Delta\mathbf{K}^o\Delta\mathbf{U}^o + \Delta\mathbf{K}^e\Delta\mathbf{U}^e = \Delta\mathbf{P}^e \tag{16}$$

Eqs.(15) and (16) have two unknowns to be solved, $\Delta\mathbf{U}^o$ and $\Delta\mathbf{U}^e$.

Through detailed derivations as shown in Appendix A, one obtains following recurrence formulae for $\Delta\mathbf{U}^o$ and $\Delta\mathbf{U}^e$

$$\mathbf{K}\Delta\mathbf{U}_k^o = -\Delta\mathbf{K}^o\Delta\mathbf{U}_{k-1}^e - \Delta\mathbf{K}^e\Delta\mathbf{U}_{k-1}^o \qquad \mathrm{k} = 1, 2, 3, \cdots \tag{17}$$

$$\mathbf{K}\Delta\mathbf{U}_k^e = -\Delta\mathbf{K}^o\Delta\mathbf{U}_k^o - \Delta\mathbf{K}^e\Delta\mathbf{U}_{k-1}^e \qquad \mathrm{k} = 1, 2, 3, \cdots \tag{18}$$

where, for the first step,

$$\mathbf{K}\Delta\mathbf{U}_0^o = \Delta\mathbf{P}^o - \Delta\mathbf{K}^o\mathbf{U} \tag{19}$$

$$\mathbf{K}\Delta\mathbf{U}_0^e = \Delta\mathbf{P}^e - \Delta\mathbf{K}^o\Delta\mathbf{U}_0^o - \Delta\mathbf{K}^e\mathbf{U} \tag{20}$$

Thus, $\Delta\mathbf{U}^o$ is iteratively obtained by

$$\Delta\mathbf{U}^o = \Delta\mathbf{U}_0^o + \Delta\mathbf{U}_1^o + \Delta\mathbf{U}_2^o + \cdots \tag{21}$$

By subtracting Eq.(9) from Eq.(10), one obtains the exact finite displacement perturbations

$$\Delta\mathbf{U} = \mathbf{U}^+ - \mathbf{U}^- = 2\Delta\mathbf{U}^o \tag{22}$$

Thus, $\Delta\mathbf{U}$ is actually obtained from the iterative solutions, $\Delta\mathbf{U}^o$ and $\Delta\mathbf{U}^e$, without using $\mathbf{U}^+$ and $\mathbf{U}^-$ which would require decomposition of $\mathbf{K}^+$ and $\mathbf{K}^-$.

Comparing Eqs.(1) and (2) and Eqs.(17 $\sim$ 21), one can see that the difference between the traditional and the modified central difference approaches is due to higher iterative terms $\Delta\mathbf{U}_k^o$ ($k \geq 1$). Those higher order terms provide valuable information about improving and estimating the sensitivity accuracy.

3. Two Error Estimators for Semi-Analytic Sensitivities

Before error estimators are defined, the computing cost involved in calculating each iterative term is discussed. Table 1 lists the total matrix operation counts for each iterative term. Adding operations are not included. For example, evaluation of $\Delta\mathbf{U}_0^o$ in Eq.(19) requires one forward and backward substitution (FBS) and one matrix multiplication (MPYAD). If the computing cost for $\Delta\mathbf{U}_0^o$ is taken as the base line, then evaluation of $\Delta\mathbf{U}_0^e$ requires additional one FBS and two MPYAD operations. Similarly, evaluation of $\Delta\mathbf{U}_1^o$ requires additional two FBS and four MPYAD operations.

3.1 Error Estimator Based on $\Delta\mathbf{U}_k^o$

A relative error based on $\Delta\mathbf{U}_k^o$ is defined as

$$\delta_k^o = \frac{\| \Delta\mathbf{U}_k^o \|}{\| \Delta\mathbf{U}_{sk}^o \|}, \quad k = 1, 2, \cdots \tag{23}$$

where

$$\Delta \mathbf{U}^o_{sk} = \Delta \mathbf{U}^o_0 + \Delta \mathbf{U}^o_1 + \cdots + \Delta \mathbf{U}^o_k \tag{24}$$

and $\| \Delta \mathbf{U}^o_k \| = \sqrt{\Delta \mathbf{U}^{oT}_k \Delta \mathbf{U}^o_k}$, $\| \Delta \mathbf{U}^o_{sk} \| = \sqrt{\Delta \mathbf{U}^{oT}_{sk} \Delta \mathbf{U}^o_{sk}}$. If δ^o_k is less than a small value ε^o, then, Eq.(24) could be taken as a sufficient approximation of $\Delta \mathbf{U}^o$ given in Eq.(21). Otherwise, one can proceed the iterative procedure.

3.2 Error Estimator Based on $\Delta \mathbf{U}^e_0$

In many cases, only an initial term $\Delta \mathbf{U}^o_0$ may be sufficient to approximate $\Delta \mathbf{U}^o$ in Eq.(21). Thus, evaluation of term $\Delta \mathbf{U}^o_1$ or the higher could be avoided. However, an error estimator which can effectively measure the sufficiency of term $\Delta \mathbf{U}^o_0$ is desirable. An relative error estimator for such a purpose is defined as

$$\delta^e_0 = \frac{\| \Delta \mathbf{U}^e_0 \|}{\| \Delta \mathbf{U}^o_0 \|} \tag{25}$$

where $\Delta \mathbf{U}^e_0$ is obtained from Eq.(20).

Since $\Delta \mathbf{U}^e_0$ contains more higher order information than $\Delta \mathbf{U}^o_0$ does, the error estimator, δ^e_0 provides valuable information about how well $\Delta \mathbf{U}^o_0$ approximates $\Delta \mathbf{U}^o$. The advantage of using δ^e_0 over δ^o_1 is to reduce the additional computing cost as much as possible.

The error estimator δ^e_0 together with δ^o_k can be used to control quality of semi-analytic sensitivities. After $\Delta \mathbf{U}^o_0$ and $\Delta \mathbf{U}^e_0$ are calculated, Eq.(25) is used to obtain δ^e_0. If $\delta^e_0 < \varepsilon^e$, then $\Delta \mathbf{U}^o_0$ is a good approximation of $\Delta \mathbf{U}^o$ and further iterations can be skipped. Otherwise, Eq.(17) is solved for δ^o_1. If $\delta^o_1 > \varepsilon^o$, it indicates that the pre-selected step size is not adequate and more terms are required to obtain accurate sensitivities. However, when the iterative process is divergent, a maximum number of iterations should be specified so as to stop meaningless calculations.

4. GENERAL ALGORITHM

Based on the idea described above, a general algorithm for the proposed approach is given below.

1. Form stiffness matrix $\mathbf{K}$ and force vector $\mathbf{P}$ and solve $\mathbf{KU} = \mathbf{P}$ for $\mathbf{U}$.

2. Build $\Delta\mathbf{K}^o\mathbf{U}$, $\Delta\mathbf{K}^e\mathbf{U}$, $\Delta\mathbf{P}^o$ and $\Delta\mathbf{P}^e$. Solve following equations for $\Delta\mathbf{U}_0^o$ and $\Delta\mathbf{U}_0^e$.

$$\mathbf{K}\Delta\mathbf{U}_0^o = \Delta\mathbf{P}^o - \Delta\mathbf{K}^o\mathbf{U}$$
$$\mathbf{K}\Delta\mathbf{U}_0^e = \Delta\mathbf{P}^e - \Delta\mathbf{K}^o\Delta\mathbf{U}_0^o - \Delta\mathbf{K}^e\mathbf{U}$$

3. Set $\Delta\mathbf{U}_{sk}^o \leftarrow \Delta\mathbf{U}_0^o$ and calculate δ_0^e. If $\delta_0^e < \varepsilon^e$, go to Step 11.

4. Set $k \leftarrow 1$.

5. Build $\Delta\mathbf{K}^o\Delta\mathbf{U}_{k-1}^e$ and $\Delta\mathbf{K}^e\Delta\mathbf{U}_{k-1}^o$. Solve the following equation for $\Delta\mathbf{U}_k^o$.

$$\mathbf{K}\Delta\mathbf{U}_k^o = -\Delta\mathbf{K}^o\Delta\mathbf{U}_{k-1}^e - \Delta\mathbf{K}^e\Delta\mathbf{U}_{k-1}^o$$

6. $\Delta\mathbf{U}_{sk}^o \leftarrow \Delta\mathbf{U}_{sk}^o + \Delta\mathbf{U}_k^o$.

7. Calculate δ_k^o. If $\delta_k^o < \varepsilon^o$, go to Step 11.

8. If $k > ITERMAX$, go to Step 12.

9. Build $\Delta\mathbf{K}^o\Delta\mathbf{U}_k^o$ and $\Delta\mathbf{K}^e\Delta\mathbf{U}_{k-1}^e$. Solve the following equation for $\Delta\mathbf{U}_k^e$.

$$\mathbf{K}\Delta\mathbf{U}_k^e = -\Delta\mathbf{K}^o\Delta\mathbf{U}_k^o - \Delta\mathbf{K}^e\Delta\mathbf{U}_{k-1}^e$$

10. Set $k \leftarrow k+1$ and go to Step 5.

11. Calculate sensitivity $\Delta\mathbf{U}_{sk}^o/\Delta\mathrm{S}$.

12. Stop.

5. PRELIMINARY RESULTS AND DISCUSSIONS

Figure 1 shows a simple cantilever beam under end moment with detailed specifications. From the basic beam theory, the deflection at the right end is given as:

$$\mathrm{V_A} = \frac{\mathrm{ML}^2}{2\mathrm{EI}} \qquad (26)$$

Taking the length of the beam L as a design variable, i.e., S = L, Then, the sensitivity of the tip deflection is

$$\frac{\partial \mathrm{V_A}}{\partial \mathrm{S}} = \frac{\partial \mathrm{V_A}}{\partial \mathrm{L}} = \frac{\mathrm{ML}}{\mathrm{EI}} = 1.0 \tag{27}$$

In order to investigate the effectiveness of the proposed approach, tip deflection sensitivities are calculated using the proposed algorithm. Four different step sizes are used in this example ($\Delta \mathrm{S} = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$). Each step size is equally distributed along the nodes on the beam model. For example, when $\Delta S = 10^{-5}$, each node is has 10^{-7} perturbation in its location for the beam model of $NE = 100$. Numerical results are presented in Tables 2 through 5. Each table is generated using one fixed step size.

Table 2 lists semi-analytic sensitivities with $\Delta \mathrm{S} = 10^{-2}$ for different numbers of elements ($NE = 1, 10, 25, 50, 100$). It shows that results from the traditional central difference scheme become less accurate when the number of elements is increased. Without corrective terms, the relative sensitivity errors range from 0.08% to 750%.

When the modified central difference approach is utilized, the magnitude of relative errors is reduced to below 1.05%. The improved accuracy is achieved by including more iterative terms in Eq.(21). These terms are included so that $\delta_k^o < \varepsilon^o = 0.05$ or $\delta_0^e < \varepsilon^e = 0.01$. When $NE = 1$, calculation of $\Delta \mathbf{U}_1^o$ and the higher is skipped owing to $\delta_0^e < \varepsilon^e$.

Sensitivities shown in Tables 3 through 5 are obtained only from the first term $\Delta \mathbf{U}_0^o$ in Eq.(21). The error estimator δ_0^e defined in Eq.(25) is calculated to ensure the accuracy of these sensitivities. It is listed in the third column. It shows that for all $\delta_0^e < 0.01$, relative sensitivity errors are well below 7%. We know from Table 1 that the additional cost for evaluating δ_0^e is one FBS and two MPYAD operations.

One may notice that in Table 5 for the case of $NE = 100$, even with $\delta_0^e = 0.025 > \varepsilon = 0.01$, the corresponding sensitivity error is very small. This is probably caused by numerical round-off errors.

The above results show that the modified central difference approach can produce very good semi-analytic sensitivities (equivalent to those using the total finite differ-

ence approach) without decomposing $\mathbf{K}^+$ and $\mathbf{K}^-$. In particular, when the step size is relatively small, the error estimator δ_0^e can be used to ensure the quality of the calculated sensitivities with minimal computing cost.

Even with a too large step size, the modified approach can use the error estimator δ_k^o to guide the iterative process to improve the sensitivity accuracy. However, as k increases, the additional computing cost also increases. As shown in Table 2, for the case of NE = 100, the additional computing cost of 12 FBS and 24 MPYAD operations is required. In practice, the iterative process can be stopped if δ_m^o is still larger than ε^o with a given number m. Then a modified step size provided a user or by the program itself can be used to continue the calculation.

6. CONCLUSIONS

1. when the shape perturbation is properly chosen, the modified central difference approach can produce as good semi-analytic sensitivities as those obtained using the total finite difference approach.

2. Even with an initially inadequate step size, The proposed approach as well as error estimators, δ_k^o and δ_0^e provides a tool to ensure the quality of calculated semi-analytic sensitivities.

3. The approach is simple and straightforward.

7. ACKNOWLEDGMENT

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APPENDIX A

Eqs.(15) and (16) in the main text are rewritten here for convenience,

$$\mathbf{K}\Delta\mathbf{U}^o + \Delta\mathbf{K}^o\mathbf{U} + \Delta\mathbf{K}^o\Delta\mathbf{U}^e + \Delta\mathbf{K}^e\Delta\mathbf{U}^o = \Delta\mathbf{P}^o \qquad (A-1)$$

$$\mathbf{K}\Delta\mathbf{U}^e + \Delta\mathbf{K}^e\mathbf{U} + \Delta\mathbf{K}^o\Delta\mathbf{U}^o + \Delta\mathbf{K}^e\Delta\mathbf{U}^e = \Delta\mathbf{P}^e \qquad (A-2)$$

Compared to $\Delta\mathbf{K}\Delta\mathbf{U}$, $\mathbf{K}\Delta\mathbf{U}^o$, $\Delta\mathbf{K}^o\mathbf{U}$ and $\Delta\mathbf{P}^o$ are higher order terms of order one, $\mathbf{K}\Delta\mathbf{U}^e$, $\Delta\mathbf{K}^o\Delta\mathbf{U}^o$, $\Delta\mathbf{K}^e\mathbf{U}$ and $\Delta\mathbf{P}^e$ are terms of order two, $\Delta\mathbf{K}^o\Delta\mathbf{U}^e$ and $\Delta\mathbf{K}^e\Delta\mathbf{U}^o$ of order three, and $\Delta\mathbf{K}^e\Delta\mathbf{U}^e$ of order four. If terms with order two or higher are ignored, Eq.(A-2) vanishes. Then, the first approximation of $\Delta\mathbf{U}^o$, $\Delta\mathbf{U}_0^o$ can be obtained using previously decomposed $\mathbf{K}$ as follows.

$$\mathbf{K}\Delta\mathbf{U}^o \approx \mathbf{K}\Delta\mathbf{U}_0^o = \Delta\mathbf{P}^o - \Delta\mathbf{K}^o\mathbf{U} \qquad (A-3)$$

Substituting $\Delta\mathbf{U}_0^o$ into Eq. (A-2) and ignoring term $\Delta\mathbf{K}^e\Delta\mathbf{U}^e$, one has the first approximation of $\Delta\mathbf{U}^e$, $\Delta\mathbf{U}_0^e$ as

$$\mathbf{K}\Delta\mathbf{U}^e \approx \mathbf{K}\Delta\mathbf{U}_0^e = \Delta\mathbf{P}^e - \Delta\mathbf{K}^o\Delta\mathbf{U}_0^o - \Delta\mathbf{K}^e\mathbf{U} \qquad (A-4)$$

Denote the errors of $\Delta\mathbf{U}_0^o$ and $\Delta\mathbf{U}_0^e$ by $\Delta\tilde{\mathbf{U}}_1^o$ and $\Delta\tilde{\mathbf{U}}_1^e$, one has true values of $\Delta\mathbf{U}^o$ and $\Delta\mathbf{U}^e$ as

$$\Delta\mathbf{U}^o = \Delta\mathbf{U}_0^o + \Delta\tilde{\mathbf{U}}_1^o \qquad (A-5)$$

$$\Delta\mathbf{U}^e = \Delta\mathbf{U}_0^e + \Delta\tilde{\mathbf{U}}_1^e \qquad (A-6)$$

Substituting Eqs.(A-5) and (A-6) into Eqs.(A-1) and (A-2), one has

$$\mathbf{K}\Delta\tilde{\mathbf{U}}_1^o + \Delta\mathbf{K}^o\Delta\mathbf{U}_0^e + \Delta\mathbf{K}^o\Delta\tilde{\mathbf{U}}_1^e + \Delta\mathbf{K}^e\Delta\mathbf{U}_0^o + \Delta\mathbf{K}^e\Delta\tilde{\mathbf{U}}_1^o = 0 \qquad (A-7)$$

$$\mathbf{K}\Delta\tilde{\mathbf{U}}_1^e + \Delta\mathbf{K}^o\Delta\tilde{\mathbf{U}}_1^o + \Delta\mathbf{K}^e\Delta\mathbf{U}_0^e + \Delta\mathbf{K}^e\Delta\tilde{\mathbf{U}}_1^e = 0 \qquad (A-8)$$

Again, ignoring relatively higher order terms in Eq. (A-7), $\Delta\mathbf{K}^o\Delta\tilde{\mathbf{U}}_1^e$ and $\Delta\mathbf{K}^e\Delta\tilde{\mathbf{U}}_1^o$ one obtains an approximate vector $\Delta\mathbf{U}_1^o$ of the $\Delta\tilde{\mathbf{U}}_1^o$ by

$$\mathbf{K}\Delta\tilde{\mathbf{U}}_1^o \approx \mathbf{K}\Delta\mathbf{U}_1^o = -\Delta\mathbf{K}^o\Delta\mathbf{U}_0^e - \Delta\mathbf{K}^e\Delta\mathbf{U}_0^o \qquad (A-9)$$

Further, one can calculate the approximate values of $\Delta\tilde{\mathbf{U}}_1^e$ by substituting $\Delta\mathbf{U}_1^o$ into Eq.(A-8) and neglecting a relatively higher order term, $\Delta\mathbf{K}^e\Delta\tilde{\mathbf{U}}_1^e$, as

$$\mathbf{K}\Delta\tilde{\mathbf{U}}_1^e \approx \mathbf{K}\Delta\mathbf{U}_1^e = -\Delta\mathbf{K}^o\Delta\mathbf{U}_1^o - \Delta\mathbf{K}^e\Delta\mathbf{U}_0^o \qquad (A-10)$$

Here, expressing

$$\Delta\tilde{\mathbf{U}}_1^o = \Delta\mathbf{U}_1^o + \Delta\tilde{\mathbf{U}}_2^o \qquad (A-11)$$

$$\Delta\tilde{\mathbf{U}}_1^e = \Delta\mathbf{U}_1^e + \Delta\tilde{\mathbf{U}}_2^e \qquad (A-12)$$

and following the similar procedure of obtaining $\Delta\mathbf{U}_1^o$ and $\Delta\mathbf{U}_1^e$, one can have approximate values $\Delta\mathbf{U}_2^o$ and $\Delta\mathbf{U}_2^e$ as follows

$$\mathbf{K}\Delta\mathbf{U}_2^o = -\Delta\mathbf{K}^o\Delta\mathbf{U}_1^e - \Delta\mathbf{K}^e\Delta\mathbf{U}_1^o \qquad (A-13)$$

$$\mathbf{K}\Delta\mathbf{U}_2^e = -\Delta\mathbf{K}^o\Delta\mathbf{U}_2^o - \Delta\mathbf{K}^e\Delta\mathbf{U}_1^e \qquad (A-14)$$

Thus, a general recurrence formula of $\Delta\mathbf{U}_k^o$ is simply derived as:

$$\mathbf{K}\Delta\mathbf{U}_k^o = -\Delta\mathbf{K}^o\Delta\mathbf{U}_{k-1}^e - \Delta\mathbf{K}^e\Delta\mathbf{U}_{k-1}^o, \quad k = 1,2,3,\cdots \qquad (A-15)$$

$$\mathbf{K}\Delta\mathbf{U}_k^e = -\Delta\mathbf{K}^o\Delta\mathbf{U}_k^o - \Delta\mathbf{K}^e\Delta\mathbf{U}_{k-1}^e, \quad k = 1,2,3,\cdots \qquad (A-16)$$

and

$$\mathbf{K}\Delta\mathbf{U}_0^o = \Delta\mathbf{P}^o - \Delta\mathbf{K}^o\mathbf{U} \qquad (A-17)$$

$$\mathbf{K}\Delta\mathbf{U}_0^e = \Delta\mathbf{P}^e - \Delta\mathbf{K}^o\Delta\mathbf{U}_0^o - \Delta\mathbf{K}^e\mathbf{U} \qquad (A-18)$$

Table 1 Total Operation Counts for Iterative Terms

Term	FBS count	MPYAD count
ΔU_0^o	1	1
ΔU_0^e	2	3
ΔU_1^o	3	5
ΔU_1^e	4	7
⋮	⋮	⋮
ΔU_k^o	2 k + 1	4 k + 1
ΔU_k^e	2 k + 2	4 k + 3
⋮	⋮	⋮

Table 2 End Deflection Sensitivities with Different Numbers of Elements

(DELB = 1. E-2)

No. of elem.	Central difference					
	Traditional		Modified			
	Sensi.	Relative error (%)	Sensi.	Relative error (%)	δ_0^e / δ_k^o (%)	Extra operations (FBS,MPYAD)
1	1.0008	0.08	1.0008	0.08	0.36 / --	(1, 2)
10	1.0751	7.51	1.0000	-0.00	0.65 / 0.01	(4, 8)
25	1.4690	46.90	0.9996	-0.04	3.47 / 0.16	(4, 8)
50	2.8757	187.57	0.9895	-1.05	5.98 / 3.07	(4, 8)
100	8.5025	750.25	0.9854	-1.46	17.4 / 2.55	(12, 24)

Table 3 End Deflection Sensitivities with Different Numbers of Elements (DELB = 1. E-3)

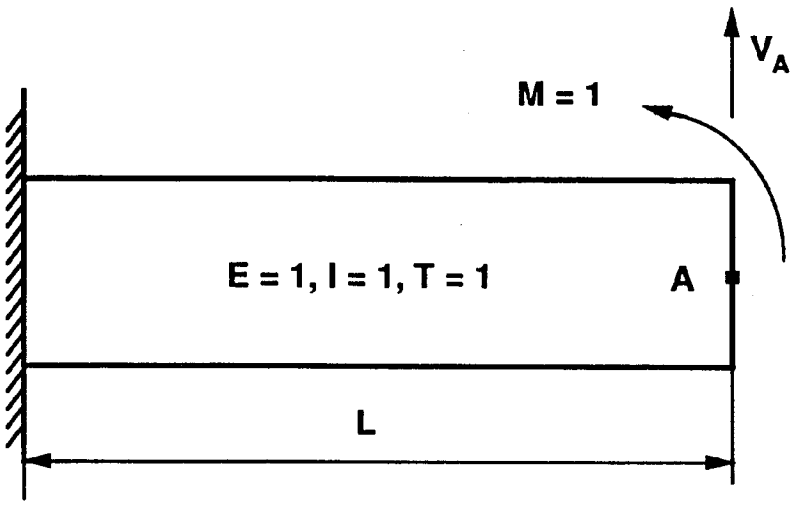
No. of elem.	Modified central difference		
	Sensi.	Relative error(%)	δ_0^e (%)
1	1.000025	0.0025	0.0375
10	1.000734	0.0734	0.0330
25	1.004475	0.4475	0.0327
50	1.017844	1.7844	0.0360
100	1.071486	7.1486	0.0425

Table 4 End Deflection Sensitivities with Different Numbers of Elements (DELB = 1. E-4)

No. of elem.	Modified central difference		
	Sensi.	Relative error(%)	δ_0^e (%)
1	1.000166	0.0166	0.0035
10	1.000172	0.0172	0.0212
25	1.000195	0.0195	0.0274
50	1.000300	0.0300	0.0302
100	1.001348	1.1348	0.2412

Table 5 End Deflection Sensitivities with Different Numbers of Elements (DELB = 1. E-5)

No. of elem.	Modified central difference		
	Sensi.	Relative error(%)	δ_0^e (%)
1	1.001358	0.1358	0.0004
10	1.001355	0.1355	0.2084
25	1.001627	0.1627	0.2508
50	1.003631	0.3631	0.3460
100	1.005467	0.5467	2.4975



$$\frac{\partial V_A}{\partial S} = \frac{\partial V_A}{\partial L} = \frac{M L}{E I} = 1.0$$

Fig. 1 A Cantilever Beam under End Moment