JOINING TETRAHEDRA TO HEXAHEDRA

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ABSTRACT

In the context of creating finite element mesh from a solid modeler, for example, ARIES/ConceptStation, the user can choose between techniques of mapped meshing and free meshing. Mapped meshing provides greater choice of elements and more control of mesh density. Mapped meshing ususally results in the most computationally efficient mesh. Free meshing provides for the most rapid production of mesh, sometimes with great sacrifice in computational efficiency. The best of both worlds would exist, if the user could choose to use mapped meshing in geometrically simple regions and free meshing in geometrically complex regions. Because currently available free meshing algorithms only provide tetrahedra, and because the most efficient (mapped) mesh consists of hexahedra, a methodology is required for joining these two non-conforming elements. This study examines a variety of methods for joining these elements, considers a multiplicity of load conditions, and demonstrates that joining with small error is possible.

INTRODUCTION

Generating finite element mesh for a three dimensional continuum is considerably more difficult than for a two dimensional continuum. Free meshing, i.e. automatic meshing free of topological constraint, is available in two dimensions for both triangles and quadrilaterals. Free meshing is commonly available in three dimensions only for tetrahedra and not for hexahedra. (This situation is expected to improve in the next couple of years as a great deal of work is going on in the National Labs and in Universities.) [1,2,3,4,5,6]

For most stress and thermal problems, the four noded tetrahedron is a poor performer. Ten noded tetrahedra, on the other hand, provide very accurate stress and temperature results for a given edge length when compared to hexahedra. Use of the ten-noded tetrahedron, however, gives rise to a larger number of degrees of freedom than a comparable hex model.

For a relatively small problem, the analyst can usually afford to mesh the entire structure with ten-noded tetrahedra and pay a bit more in CPU cost. For a large problem, computer resources may be an important issue. The best compromise occurs when the structure can be divided into sub-regions in such a way as to segregate the geometrically complex regions. These can be meshed with tetrahedra, while the remaining geometrically simple regions can be meshed with hexahedra. See Fig 1. In this way, the size of the model remains manageable and the time to mesh can be drastically reduced.

PROBLEM DEFINITION

Two issues need to be resolved before it is practical to wed mapped mesh to free mesh (hexahedra to tetrahedra) in a three-dimensional continuum. The first issue is that the <u>corner nodes should match across the interface</u>. Some free mesh generators, for example, are incapable of matching the regular spacing of a mapped mesh. The second issue is that the tetrahedra and hexahedra are <u>non-conforming at their interface</u>, i.e. they are not C^0 continuous. To illustrate this, consider two tetrahedra whose triangular faces join the quadrilateral face of a hexahedron as illustrated in Fig 2. The center point of the diagonal, point A, is coincident with the center of the quadrilateral face before deformation. Point A on the tetrahedra will displace only as a function of points 1 and 3. The coincident point of the quadrilateral face will displace as a function of all four bounding points. Therefore, the deformed geometry will in general be discontinuous at the element interface.

The first issue, that of matching corner nodes across the interface, will be a function of the algorithm employed by the mesh generation software. ARIES/ConceptStation, which uses an Octree algorithm, permits the seeding of the volumetric free mesh by using either a mapped mesh or a free mesh on bounding faces. Similarly, software that uses the Delaunay algorithm permits seeding by either mapped or free meshing.

The second issue, that of compatibility (C^0 continuity), is the main subject of this paper.

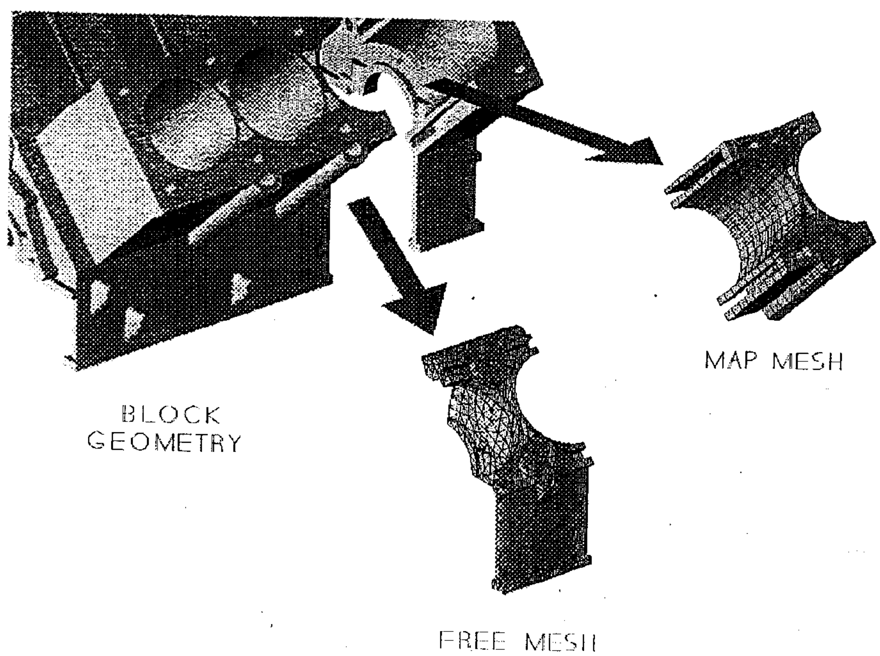

FIGURE 1

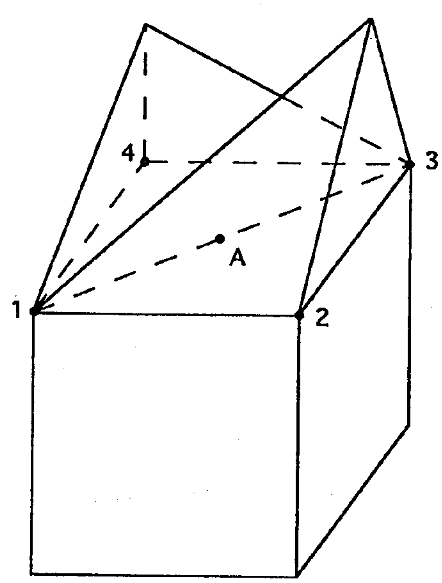

Figure 2

ANALYSIS

Because MSC/NASTRAN permits the omission of any combination of edge nodes on a higher order element, it is possible to join hexahedra to tetrahedra in a great number of ways. For example, one possibility is to drop the mid-diagonal node (node A of Fig 2) so as to join CTETRA(9)s to CHEXA(20)s. This provides continuity at the corners and mid-edges. Another possibility is to omit all three mid-edge nodes at the interface and join CTETRA(7)s to CHEXA(8)s. We tested six combinations as follows:

CTETRA(4) to CHEXA(8)
CTETRA(9) to CHEXA(12)
CTETRA(9) to CHEXA(20)
CTETRA(10) to CHEXA(8)
CTETRA(10) to CHEXA(12)
CTETRA(10) to CHEXA(20)

The CTETRA(10)s, which of course retained their mid-diagonal nodes, had their mid-diagonal nodes constrained to displace as a function of the four surrounding corner nodes by means of multi-point constraint functions (MPCs). This has the effect of smoothing out the displacement field across the element interface. The MPC prescribes the displacement of the mid-diagonal node as a weighted sum of the displacements of the four surrounding corner nodes. The weighting factors can be calculated from the solution of four non-linear simultaneous equations derived from consideration of the element shape functions and the weighting factor identity equation. A good discussion of this is found in Reference [7], which suggests an iterative solution to the non-linear equations. An exact solution for these equations is given in the appendix to this paper. In our experience, the exact solution provides a more robust and efficient procedure than the iterative scheme that we tried. As a trivial example of the weighting functions, when the face of the CHEXA element is rectangular, the weighting functions are 0.25 for each of the four nodes bounding point A.

The six combinations listed above were tested for a bar, a prismatic structure whose cross-section is an equilateral triangle. The bar is divided into two halves at the midpoint of its length. One half of the geometry is filled with CTETRA elements and the other half with CHEXA elements. See Fig 3. Stresses at the central plane where the tetrahedra join to the hexahedra are measures of performance. Theoretical stresses at this interface are of course known. The bar was subject to tension/compression, torsion, bending, free thermal growth, and a large rigid body displacement, all separately applied. Those five loads or displacements cover the range of conditions that could be expected.

In addition to the six combinations of the list, three separate models consisting solely of CHEXA(8), CTETRA(10), and CTETRA(4) were tested to provide a basis for comparison to the heterogeneous models. Note that the triangular cross-section produces elements that are somewhat distorted. We consider the amount of distortion to be typical of that seen in real models.

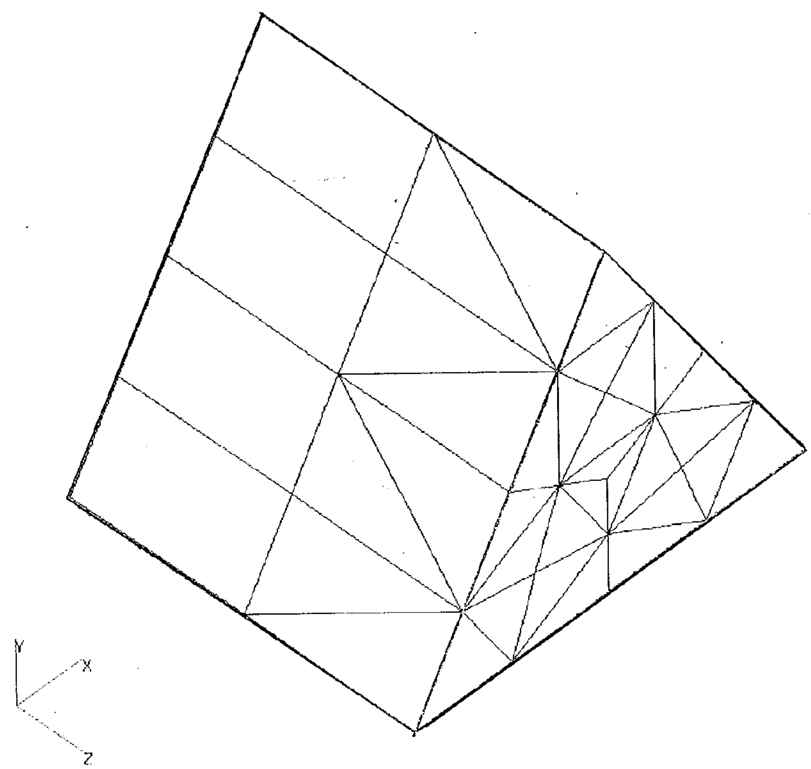

FIGURE 3

DISCUSSION

Table 1 summarizes the results of 50 tests on the various elements and loads or displacements. Note that the TENSION test produces EXACT results for all three of the homogeneous element sets. This confirms that those elements pass the patch test. Note also that the THERMAL test and the RIGID BODY DISP. test produce negligible error for all cases except those where MPCs are used. The CTETRA(4) element produces very large errors in the TORSION and BENDING tests. (These errors would be reduced for a finer mesh.) It's also interesting that the CTETRA(10) outperforms the CHEXA(8) in torsion, while their roles are reversed for bending.

Note that the von Mises stresses associated with the THERMAL case and the RIGID BODY DISP. case are not negligible for the heterogeneous models. This fact apparently arises due to the precision, or lack thereof, associated with the MPC coefficients. The normal eight column format of the MSC/NASTRAN input deck, after providing one column for the sign, one column for a leading integer, and one column for the decimal point can leave as few as five digits. This situation is alleviated by switching to the sixteen column format. The results from that increase in precision are labeled (WIDE MPC) in the table.

The results reported for CHEXA(12) and CHEXA(20) are not based on weighting factors obtained for all eight surrounding nodes. That requires an extension of the procedure given in the appendix. Presumably the use of all eight surrounding nodes would provide somewhat lower error.

CONCLUSION

The results of Table 1 demonstrate that joining CHEXA(8) to CTETRA(10) can be accomplished. For the benchmark problem, the error in stress is less than 0.5% for tension or bending and less than 7% for torsion. (The corresponding stress error for torsion is 10% for an all-CHEXA(8) model). The recommended technique is to use multipoint constraints (MPC's) for the mid-edge nodes of the CTETRAs which lie in the plane of joining. The nodes between the corners use weighting factors of 0.5 since they are halfway between the corners. The node which corresponds to point A of Fig 2 uses four weighting factors as suggested in the appendix.

Table 1

Results of Benchmark Tests on Triangular Prism. MSC/NASTRAN V66

	TENSION	TORSION	BENDING	THERMAL	RIGID BODY DISP.
HEXA8	EXACT	TORQUE: 10% ERR.	STRESS: 1.5% ERR.	σ_{VM} OF ORDER 10E-8	σ_{VM} OF ORDER 10E-6
TET10	EXACT	TORQUE: 1.5% ERR.	STRESS: 4.3 % ERR.	σ_{VM} OF ORDER 10E-8	σ_{VM} OF ORDER 10E-6
TET 4	EXACT	TORQUE: 256% ERR.	STRESS: 56% ERR.	σ_{VM} OF ORDER 10E-9	σ_{VM} OF ORDER 10E-8
HEXA8/ TET4	DISP.: 2.3% ERR. STRESS: 9.82% ERR.	TORQUE: 58.5 % ERR.	STRESS: 23.% ERR.	σ_{VM} OF ORDER 10E-6	σ_{VM} OF ORDER 10E-8
HEXA12/ TET9	DISP.: 4.6% STRESS: 21 %	TORQUE: 1.5% ERR.	STRESS: 16.1% ERR.	σ_{VM} OF ORDER 10E-6	σ_{VM} OF ORDER 10E-8
HEXA20/ TET9	DISP: +6.3% STRESS: +33.7%	TORQUE: -1.2%	STRESS: +6.63%	σ_{VM} OF ORDER 10E-8	σ_{VM} OF ORDER 10E-6
HEXA8/ TET10 (MPC)	DISP. EXACT STRESS: 0.47% ERR.	TORQUE: 6.5%	STRESS: 0.43% ERR.	σ_{VM} 15 psi	σ_{VM} 2270 psi
HEXA8/ TET10 (WIDE MPC)	DISP. EXACT STRESS: 0.48% ERR.	TORQUE: 6.5%	STRESS: 0.43% ERR.	σ_{VM} 0.035 psi	σ_{VM} 5.26 psi
HEXA12/ TET10 (MPC)	DISP: +3.% STRESS: +17.5%	TORQUE: -1.%	STRESS: +14.7%	σ_{VM} 13 psi	σ_{VM} 2180 psi
HEXA20/ TET10 (MPC)	DISP: +3.% STRESS: +25.6%	TORQUE: -1.8%	STRESS: +12.8%	σ_{VM} 13.3 psi	σ_{VM} 2149 psi

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Appendix
Isoparametric Interpolation of Quadrilaterals

Nomenclature:

x_1, y_1 coordinates of point 1 of Fig 2,
x_2, y_2 coordinates of point 2 of Fig 2,
X_3, Y_3 coordinates of point 3 of Fig 2,
x_4, y_4 coordinates of point 4 of Fig 2.

$x'(x)$ local x
$y'(y)$ local y

A,B,C parameters whose values depend on nodal geometry
D(x,y),E(x,y),F(x,y) parameters whose values depend on nodal geometry and on local x and local y.

r(x,y) and s(x,y) are the natural (transformed) coordinates of x,y.

$h_1(r,s) \ldots h_4(r,s)$ are the weighting coefficients for the MPC.

u(r,s)...w(r,s) are the displacements of point at r,s as function of nodal displacements.

Procedure:

For the point, which lies on the tetrahedron mid-edge and is to be constrained, determine its local coordinates x'(x) and y'(y). Determine the parameters A,B,C,D(x,y),E(x,y), and F(x,y). Evaluate r(x,y) and s(x,y). Calculate $h_1(r,s)$,$h_2(r,s)$,$h_3(r,s)$and $h_4(r,s)$. These are the coefficients for the MPC card.

Equations:

$$x'(x) = x - \frac{1}{4} (x_1 + x_2 + x_3 + x_4)$$

$$y'(y) = y - \frac{1}{4} (y_1 + y_2 + y_3 + y_4)$$

$$A = \frac{1}{2}\left[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3) \right]$$

$$B = \frac{1}{2}\left[(x_1 - x_2)(y_3 - y_4) - (x_3 - x_4)(y_1 - y_2) \right]$$

$$C = \frac{1}{2}\left[(x_1 - x_4)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_4) \right]$$

$$D(x,y) = x'(x)(y_1 - y_2 + y_3 - y_4) - (x_1 - x_2 + x_3 - x_4)\,y'(y)$$

$$E(x,y) = x'(x)(y_1 + y_2 - y_3 - y_4) - (x_1 + x_2 - x_3 - x_4)\,y'(y)$$

$$F(x,y) = -x'(x)(y_1 - y_2 - y_3 + y_4) + (x_1 - x_2 - x_3 + x_4)\,y'(y)$$

$$r(x,y) = \frac{E(x,y)}{A - D(x,y)}\left[\frac{2}{1 + \sqrt{1 + \dfrac{4\,B\,E(x,y)}{\left[A - D(x,y)\right]^2}}} \right]$$

$$s(x,y) = \frac{F(x,y)}{A + D(x,y)}\left[\frac{2}{1 + \sqrt{1 + \dfrac{4\,C\,F(x,y)}{\left[A + D(x,y)\right]^2}}} \right]$$

$$h_1(r,s) = \frac{1}{4}(1 + r)(1 + s) = \frac{1}{4}(1 + r + s + rs)$$

$$h_2(r,s) = \frac{1}{4}(1 - r)(1 + s) = \frac{1}{4}(1 - r + s - rs)$$

$$h_3(r,s) = \frac{1}{4}(1 - r)(1 - s) = \frac{1}{4}(1 - r - s + rs)$$

$$h_4(r,s) = \frac{1}{4}(1 + r)(1 - s) = \frac{1}{4}(1 + r - s - rs)$$