EULER BUCKLING

Kevin E. Arden

Newport News Shipbuilding
4101 Washington Avenue
Newport News, Virginia 23607

ABSTRACT

This paper is intended to investigate the accuracy of MSC/NASTRAN's Solution 105 for use in calculating linear elastic (Euler) buckling modes.

Column buckling, panel buckling, and stiffened panel buckling is analyzed using Euler equations and Solution 105. Comparisons and modeling recommendations are made for each type of structure.

Solution 105 provides excellent results for Euler type buckling. Panel buckling requires the use of an adequate number of elements. A convergence plot shows that four QUAD4 elements per half sine wave are necessary for accurate results.

The opinions expressed herein are those of the author and do not necessarily reflect those of Newport News Shipbuilding and Dry Dock Company.

Euler Buckling

Introduction

The Euler buckling load is the load for which an ideal structure will first become unstable and buckle if slightly perturbed from its equilibrium position.

Due to eccentricity of the column and load, inelastic action, and residual stresses, the ultimate load will be less than the Euler buckling load.

Generally, the ultimate load is the load of concern, but the Euler buckling load is helpful information. The linear elastic buckling load is easily calculated and can give an idea of the type and pattern of ultimate failure.

This paper will compare solution 105 (linear elastic buckling) to the Euler buckling equations for different types of structures. Element type and mesh density will be determined for best results.

BEAM BUCKLING

Basic Theory

The maximum load that a column can carry depends on many factors. Eccentricity of the column and load, inelastic action, residual stress, and other factors affect the maximum load. The Euler buckling load is an ideal load which only accounts for a guided load and various end restraints.

The Euler buckling load (Reference (a))

$$P_{cr}=\frac{\pi^2 EI}{L_e^{\,2}} \qquad (1)$$

is the load for which this ideal column will first have a buckled mode shape. It is the eigenvalue in the solution to Euler's differential equation

$$\frac{d^2x}{dy^2}+\frac{Px}{EI}=0\,. \qquad (2)$$

Bifurcation Buckling by the Eigenvalue Method

The equilibrium equation

$$F=(K+K_G)\,\Delta\,, \qquad (3)$$

where K_G is the geometric stiffness matrix, can be expressed in incremental form. For an increment of load F_i and a corresponding increment of deflection, equation (3) is still valid if K_G is evaluated for the structure's current state.

The critical condition is a condition where there can be a non-zero deflection with no increase in load, setting $F_i = 0$, equation (3) is now

$$(K+K_G)\,\Delta_i=0\,. \qquad (4)$$

Since the deflection increment is defined as non-zero the mathematical criteria for buckling is

$$\det(K+K_G)=0. \qquad (5)$$

The load vector can be defined as a scalar multiple of a representative (applied) load.

If F_O = the applied load then,

$$F=\lambda F_O \quad , \quad K_G=\lambda K_{G_O} \; .$$

Using this in equation (4) gives,

$$(K+\lambda K_{G_O})\,\Delta=0. \qquad (6)$$

The critical buckling load is the lowest eigenvalue of equation (6) times the applied load,

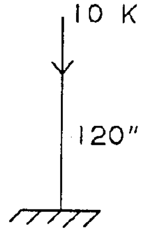
$$F_{cr}=\lambda_1 F_O. \qquad (7)$$

The following pages show the solution for the beam in Figure 1, page 5, using the eigenvalue method (6), Euler's equation (1), and MSC/NASTRAN Solution 105.

Euler's equation gives a lambda of 10.28 and the other two solutions yield 10.36. As was expected, Solution 105 gives excellent Euler buckling results for beam elements.

BUCKLING SOLUTION BEAM ELEMENT

(LOAD IS GUIDED)

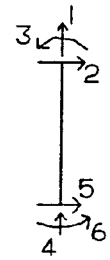


$A = 5\ IN^2$

$I = 20\ IN^4$

$E = 30E6\ PSI$

FIGURE 1



DOF ASSIGNMENT

(Reference (c))

$$K_{FF} = \begin{bmatrix} AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L \end{bmatrix}, \quad F_F = \begin{bmatrix} -10000 \\ 0 \\ 0 \end{bmatrix},$$

$$K_{FF} = \begin{bmatrix} 125X10^4 & 0 & 0 \\ 0 & 4166.67 & 25X10^4 \\ 0 & 25X10^4 & 2X10^7 \end{bmatrix}, \quad F_F = K_{FF} X_F, \quad X_F = K_{FF}^{-1} F_F,$$

$$K_{FF}^{-1} = \begin{bmatrix} 8X10^{-7} & 0 & 0 \\ 0 & 9.6X10^{-4} & -12X10^{-6} \\ 0 & -12X10^{-6} & 2X10^{-7} \end{bmatrix} \quad X_F = \begin{bmatrix} -.008 \\ 0 \\ 0 \end{bmatrix}$$

$\Delta = PL/AE = -.008$

MATRIX METHOD IS
EXACT FOR BEAMS

THE GEOMETRIC STIFFNESS MATRIX IS

$$K_G = \begin{bmatrix} 1.2F_X/L & 0.1F_X \\ 0.1F_X & (2/15)F_X L \end{bmatrix}, \quad F_X = -10000 \text{ LBS}, \quad L = 120''$$

Reference (b)

$$K_G = \begin{bmatrix} -100 & -1000 \\ -1000 & -160000 \end{bmatrix}, \quad K = \begin{bmatrix} 4166.6 & -25\times10^4 \\ -25\times10^4 & 2\times10^7 \end{bmatrix}$$

$$[K + \lambda K_G]X = 0$$ EIGENVALUE METHOD

SOLUTION $\lambda_1 = 10.4$, $\lambda_2 = 134.1$

EULER'S FORMULA

$$P = \frac{\pi^2 EI}{L_e^2} \qquad \lambda_1 = 10.281$$

MATRIX SOLUTION IS (+0.749)% HIGHER

MSC/NASTRAN Solution 105 Input Deck

```
SOL 105
TIME 5
CEND
SUBCASE 1
SPC=1
LOAD=1
DISP=ALL
SUBCASE 2
METHOD=10
SPC=1
LOAD=1
DISP(PLOT)=ALL
BEGIN BULK
GRID,1,0,0.,0.,0.
GRID,2,0,0.,120.,0.
CBAR,1,1,1,2,1.
MAT1,1,30.E6, ,.3
PBAR,1,1,5.,20.,20.,1.
FORCE,1,2,0,-10000.,0.,1.,0.
SPC,1,1,123456
SPC,1,2,345
EIGRL,10,0.,150.,2
ENDDATA
```

Solution

$\lambda_1=10.358$ $\lambda_2=134.09$

$\Delta_{static}=.008$

Results are the same as before.

PANEL BUCKLING

The Euler type buckling formula for a simply supported plate under uniform uniaxial load is, (Reference (b)),

$$\sigma_{cr}=K\frac{\pi^2 D}{b^2 t} \qquad (8)$$

where,

$$D=\frac{Et^3}{12(1-v^2)} \quad , \quad K=[\frac{mb}{a}+\frac{a}{mb}]^2$$

for n=1, where m,n define the mode's shape (number of waves each direction).

K=4.0 is the lowest possible mode for a rectangular simply supported plate.

The following page shows the Euler solution and Solution 105 results for the panel in Figure 2, page 7.

Given that plate elements are not exact formulations, a certain mesh refinement is required for an acceptable solution.

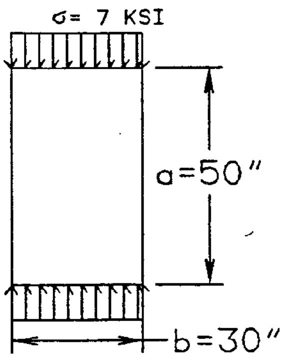


FIGURE 2

ALL EDGES SIMPLY SUPPORTED

$$\sigma_{CR} = K \frac{\pi^2 D}{b^2 t}$$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

$$K = \left(\frac{mb}{a} + \frac{a}{mb}\right)^2$$

$t = 0.25''$
$E = 30{,}000{,}000$ psi
$\nu = 0.3$

D = 42925.824

FOR K=4.0 (LOWEST VALUE)

σ_{CR} =7531.75 psi

λ = 1.076 $\quad \lambda = \dfrac{\sigma_{cr}}{\sigma}$

MSC/NASTRAN SOL 105

EIGENVALUES

MESH SIZE	QUAD4	QUADR
8X8	1.084	1.117
6X6	1.079	1.144
4X4	1.086	1.209
2X2	0.468	0.449
1X1	0.107	0.006

Figure 3 shows the convergence rate of both the QUAD4 and QUADR element. Based on Figure 3 it is concluded that four QUAD4 elements are adequate to model the half sine wave of the buckled mode shape. The QUAD4 and the theoretical curves in Figure 3 appear to converge at four to six elements. If the QUADR element is used for buckling, twice as many elements appear to be necessary.[1]

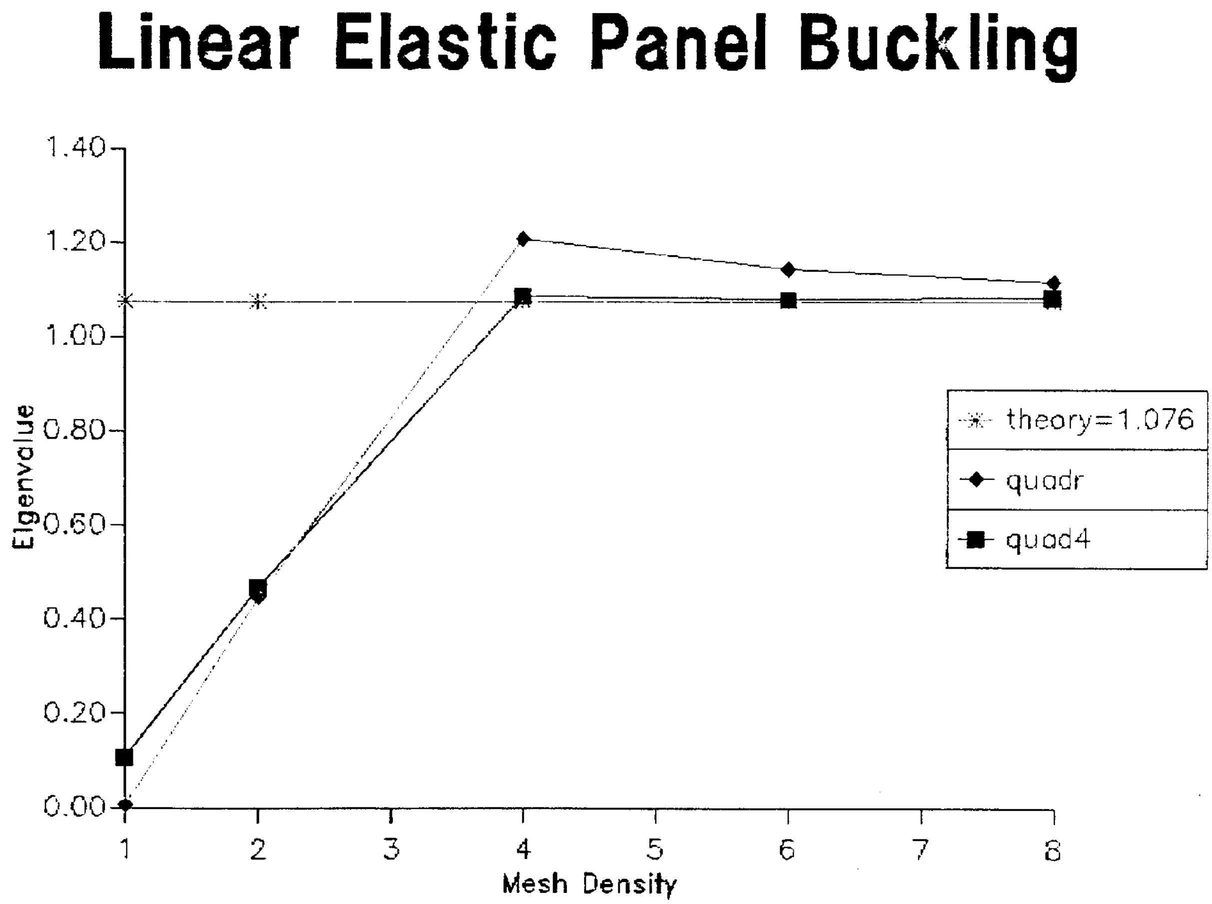
Figure 3 shows convergence from the low side for this problem. Experience has shown that for more complex models, convergence is from the high end. Eigenvalues 7 to 8 times higher than expected have been seen when there are less than four elements on the half sine wave.

Poor mesh density is easily seen from plotting the mode shapes. The mode shapes will appear to be jagged or pointed.[2] In summary, an adequate amount of elements provide excellent results for Euler buckling.

---

[1]Much worse results have been seen when using the QUADR element in complex models. Hence, the QUADR is not recommended for buckling.

[2]Sometimes referred to as mountain peaks since the panels deform to a sharp point instead of smooth curves.



Linear Elastic Panel Buckling
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
Eigenvalue
1
2
3
4
5
6
7
8
Mesh Density
theory=1.076
quadr
quad4

FIGURE 3

STIFFENED PANEL BUCKLING

Stiffened panels can buckle in two ways, overall buckling and local buckling. In overall buckling the stiffeners buckle along with the plating and in local buckling the panels buckle first.

Figure 4 shows a stiffened panel under an in-plane load. Panel buckling is analyzed by assuming each panel is simply supported and equation (8) is used. Overall buckling is analyzed by assuming an effective width of plate acts with the beam forming a column. The width of plate that acts with the beam is based on shear lag. Load is carried by the plate and stiffeners. In overall buckling a portion of the plate will act with the stiffener by a transfer of shear. The further from the stiffener the plate is the less effective it is due to shear lag. The effective width used in the hand calculations is

$$b_p = 2.0\, t_p \sqrt{\frac{E}{\sigma_y}}$$

where;

b_p=*effective width of plate*
t_p=*plate thickness*
E=*Modulus of Elasticity*
σ_y=*Yield stress*

Equation (1) is used to evaluate the overall (columns buckling) strength.

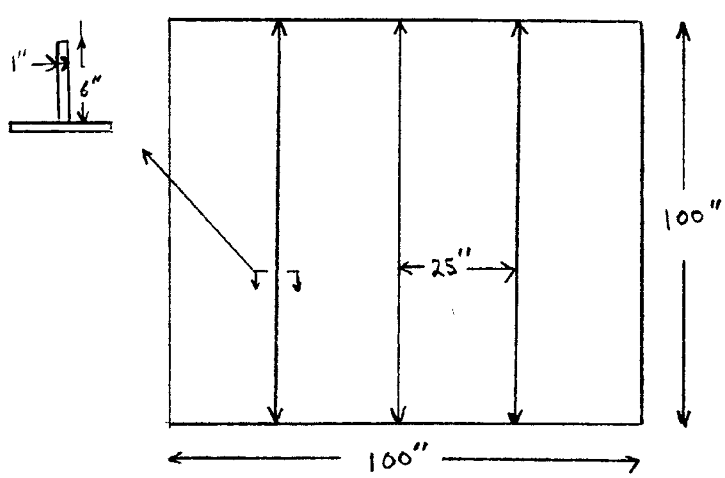


Figure 4

Local Buckling (Panel)

Equation (8) yields a panel buckling eigenvalue of 1.0. Figure 5 shows the solution 105 results for the stiffened panel. Even though there are 6 elements per sine wave in the athwartship direction there are only two in the longitudinal direction. Also, equation (8) is for simply supported panels and the stiffeners provide some rotational restraint. For a fixed boundary the theoretical solution is 1.6. The bounds of the theoretical solution are 1.0 and 1.6. An FEA model result of 1.008 including the rotational restraint of the stiffeners appears to be an acceptable answer. The stiffeners provide little rotational support and the panels buckle as if they are simply supported.

Overall Buckling (Column)

Figure 6 shows the MSC/NASTRAN result for overall buckling to be 5.042. This is 0.04% higher than the theoretical solution given by equation (1). Figure 6 shows the first mode that involves the stiffener and is considered the overall buckling condition predicted using the effective width of plate. It appears that if enough elements are used the QUAD4 can model the shear lag effect. Hence, building a separate of just the beam and the effective width of plate should yield the same result. Although, with the entire model it may be difficult to find the first overall mode. A separate model of just the effective column would yield the overall buckling mode in one eigenvalue.

The above results show that solution 105 compares very well to Euler's buckling equations when appropriate element density is used.

STIFFENED PANEL BUCKLING

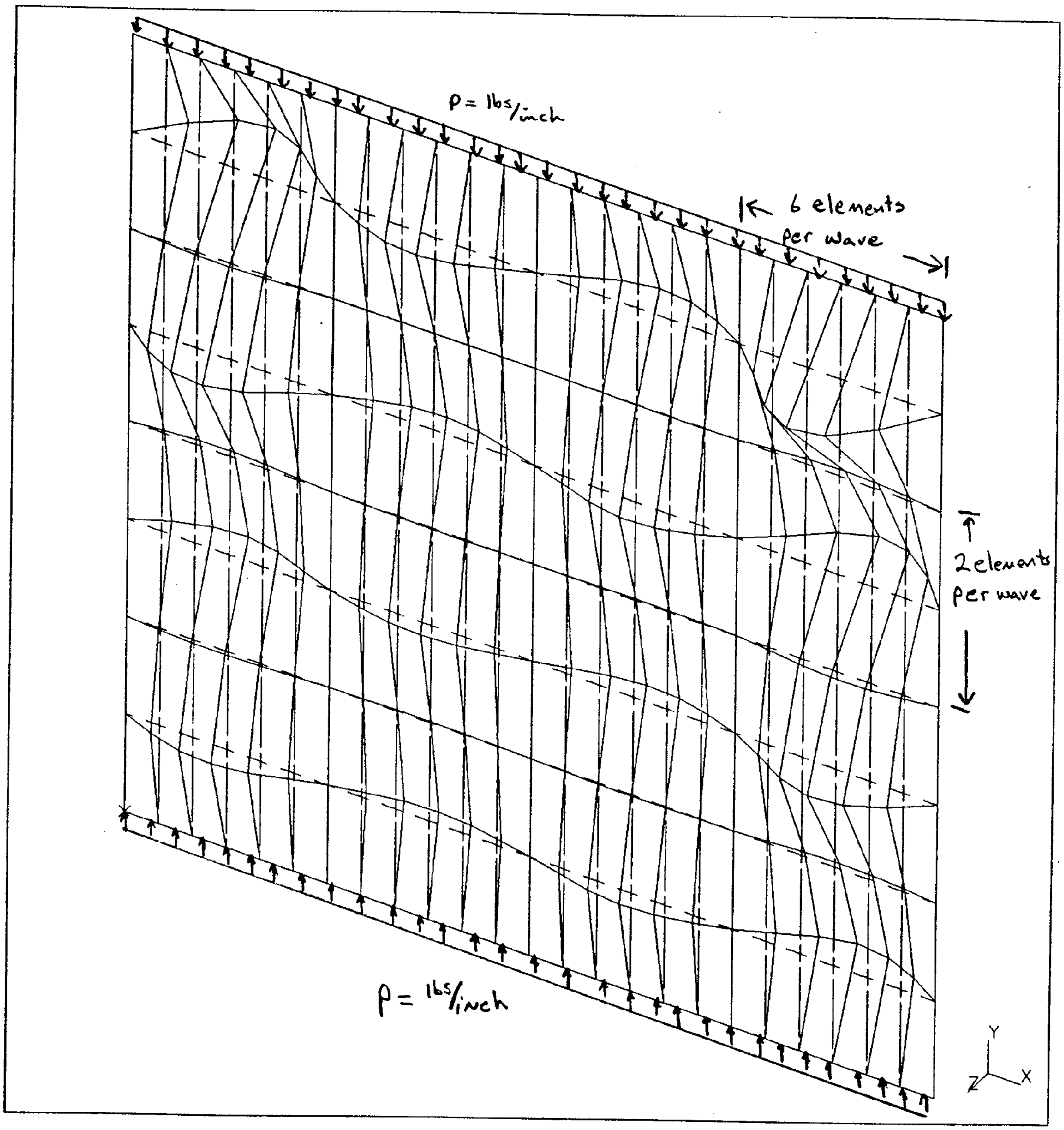


FIGURE 5
LOWEST MODE SHAPE
EIGENVALUE = 1.008

STIFFENED PANEL BUCKLING

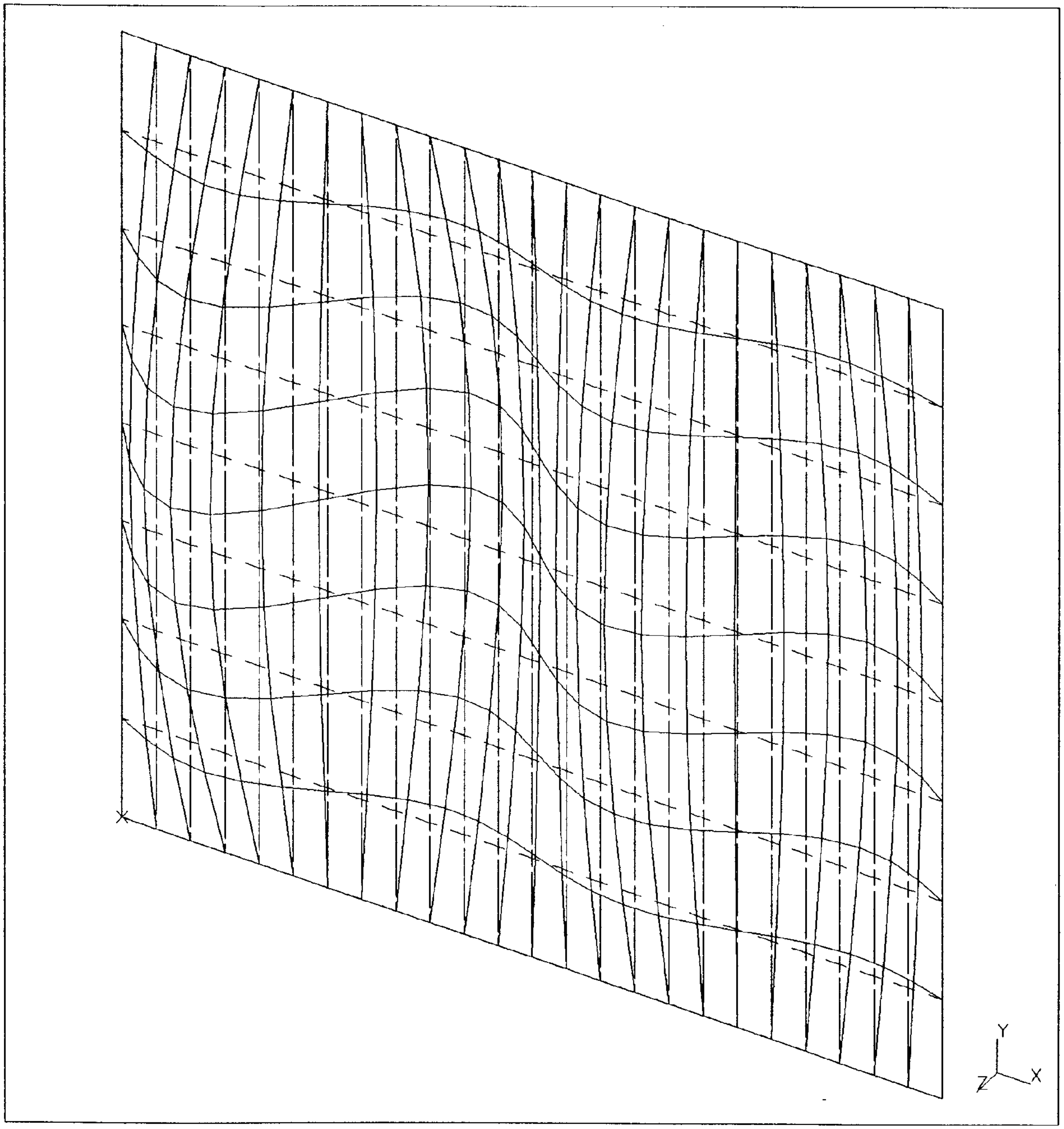


FIGURE 6

LOWEST STIFFENER MODE SHAPE

EIGENVALUE = 5.042

Conclusions

The results of this paper show that solution 105 yields excellent results for linear elastic buckling.

Beam elements appear to give exact results except when mixed with plate elements and an effective width of plate acts as a flange. In this case, four elements along the half sine wave in the primary direction provides accurate results for overall buckling of the plate and stiffeners.

Panel buckling can be accurately predicted using QUAD4 elements with at least four elements per half sine wave of the mode shape. The number of elements in the secondary direction seems to be less important. The quality of the mode shape is a good indication of accuracy. Jagged deformed plots are questionable, smooth curves along the sine waves are preferable.

The QUADR element should not be used for buckling. In this report the QUADR is converging much slower than the QUAD4, and other analyses, not documented in this report, have yielded poor results.

Reviewer Comments

The bar element model will converge from 10.358 to 10.282, if the mesh is increased to 3 elements. The theoretical value is 10.281.

The QUADR element should not be used for buckling for the following reasons:

1) It does not have a geometric stiffness matrix.

2) Consistent loads, as opposed to point loads, are required. In other words, equivalent end moments in addition to forces need to be included.

REFERENCES

(a) Beer, Ferdinand P. and E. Russell Johnston, Jr., Mechanics of Materials, McGraw-Hill Book Company, 1981.

(b) Hughes, Owen F., Ship Structural Design, A Rationally-Based, Computer-Aided, Optimization Approach, John Wiley & Sons, 1983.

(c) Baufait, Fred W., William Rowan, Jr., Peter G. Hoadley, and Robert M. Hackett, Computer Methods of Structural Analysis, Prentice Hall, 1970.