Production Oriented Nonlinear Analysis of Solids and Structures

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Abstract

Requirements for modeling nonlinear effects in routine analysis applications have grown to the point where "general purpose" finite element-based programs are expected to offer significant nonlinear modeling capabilities. One result of this market demand has been this year's announcement of the establishment of a long term relationship between The MacNeal-Schwendler Corporation and Hibbitt, Karlsson Sorensen, Inc. ("HKS"), whereby MSC will package a substantial set of the capabilities offered by HKS's ABAQUS/Standard program with MSC/ARIES, to supplement the nonlinear capabilities of MSC/NASTRAN and MSC/DYTRAN for applications in solid and structural analysis. MSC will provide full support of these capabilities, as it does for its other analysis products.

Nonlinear effects introduce a broad range of issues which might deter the analyst who is unfamiliar with this type of problem from trusting such modeling as a basis for achieving design goals and schedules. Once nonlinearity is introduced into a model, uniqueness and stability of the solution may be (and often are) lost, and issues of convergence, choice of nonlinear solution algorithm, etc. must be considered. Nevertheless, the analyst may have no choice but to face up to these problems: he cannot analyze a design for certain events, or design the manufacturing process to create a product, without considering nonlinearity. The viewpoint taken in this paper is that, with mature software such as the ABAQUS-based products that MSC now offers, some nonlinear effects of practical importance can be modeled on a routine, production, basis. The spectrum of difficulty ranges from such cases all the way to problems that are still research topics. One purpose of this paper is to suggest what level of difficulty might be anticipated in modeling various nonlinear effects that are commonly encountered, thus providing some guidance to the analyst in determining the extent to which expertise is needed in order to utilize nonlinear analysis software.

Background

MSC's new nonlinear analysis products are based on HKS's ABAQUS/Standard program. ABAQUS/Standard is a general purpose finite element program which has always emphasized nonlinear applications. The program began as a "clean sheet of paper" design in 1978. Throughout its history ABAQUS/Standard has been developed by HKS as a commercial application, so that its success has been based on meeting the demands of industrial use. The software must perform relevant simulations—robustness and broad application coverage are, therefore, key issues. Practical problems are generally complex, so that algorithms must work in combination, over the widest range of models. We have therefore tended to concentrate on "old fashioned" (proven) methods.

As nonlinear analysis becomes accepted as a routine activity within the design process, HKS's view is that nonlinear analysis capabilities must be integrated into general analysis systems. HKS's relationship with MSC is one approach to meeting this need. MSC/ARIES provides a geometry-based analysis environment, within which the user will access a subset of ABAQUS/Standard directly, without going through any translators. In this sense HKS is acting as a component supplier to MSC, as it does to other vendors of general analysis systems. The breadth of the market for nonlinear analysis is sufficient to create demand for HKS to offer its software directly (which it does, on a worldwide basis) and as components in other systems. And the complexity and diversity of nonlinear analysis problems provides the opportunity for HKS to continue its focus on developing software to address them.

Typical applications

A brief listing of some typical applications where HKS has seen customers using nonlinear analysis provides some understanding of the breadth of such usage. The list below is organized by industry. It is incomplete, but includes enough entries to offer a sense of the range of problems to which HKS's products are routinely applied.

Surface vehicles (mostly passenger cars, but also trucks, trains, earth moving equipment):

- suspension system design (large motion of almost rigid components with deformable bushings; design criteria typically include fatigue and severe events).
- engine mounts (rubber, metal and fluid; large strain).
- tires (reinforced rubber enclosing compressible gas, with contact).
- weather-stripping (rubber with contact).
- crash, roll-over (metal visco-plasticity, large deformation of structural members, complex contact). Such analyses are most often done with explicit dynamics programs, such as MSC/DYTRAN and ABAQUS/Explicit.
- seat design (foam rubber, finite strain, contact, inelastic vibration).
- glass windshield, light lens manufacture (high temperature viscoelastic response of glass, with contact).
- gear design (contact; fatigue).
- brake design (thermal fatigue, contact, strongly coupled thermal and mechanical effects).
- engine components (contact, thermal fatigue).

- convertible top (large displacements of membranes and mechanisms).
- sheet metal forming (contact, friction, metal plasticity, moderately large strain in shells, springback). This application seems most effectively tackled with a combination of explicit and implicit analysis.
- plastic component manufacturing (post-molding springback, buckling).

Defense

- blast loading.
- solid rocket motors (viscoelastic propellant, seals, contact).
- submarines (pressure hull collapse, underwater shock, welding, fracture mechanics, acoustics, piezoelectric effects).

Nuclear power safety

- pressure vessels (high temperature metal plasticity and creep, fracture mechanics).
- containment structures (reinforced concrete under extreme loads).
- waste storage (geotechnical materials, flow through porous medium, thermal effects).
- piping systems subject to severe earthquake events (plasticity in the pipes, complex nonlinearities in snubbers, relatively long response times).

Oil extraction

- flow through collapsing porous medium (geotechnical materials, coupled pore fluid flow and large deformation).
- threaded connectors (metal plasticity, contact with large sliding).
- well liner buckling (metal plasticity, geotechnical material modeling, sliding contact, unstable large displacements).
- drill string buckling in directional drilling (complex buckling cases, where higher order buckling modes are often critical).

Offshore oil installations

- fatigue of steel jackets (inelastic fracture mechanics).
- pile/soil interaction; severe foundation conditions (inelastic geotechnical material modeling for cyclic loading).
- concrete gravity platform design (heavily loaded reinforced concrete; construction, installation, fatigue and accident analysis).
- remote installation of underwater pipelines (large displacements of very slender members).
- "floaters" (tension leg platforms). Large motions of flexible systems subjected to wave and wind loading.
- fire hazard studies.
- removal of unneeded platforms (demolition).

Other civil engineering

- tunneling (inelastic deformation of soils, rocks, concrete, rock bolts; complex sequencing of excavation and installation; complex geometries).

- design of elevated highways and other structures subject to large energy earthquakes.

Electronic components

- thermal fatigue of solder joints (creep/plasticity, complex contact, fracture mechanics).

High volume consumer goods

- package (bottle) design (paper laminates, paperboard, plastic, aluminum. Collapse of liquid filled structure; manufacturing problems).
- child-proof seals (contact, inelastic deformation, instability).
- drop tests (dynamics, contact).
- diapers (partially saturated flow through highly porous, deforming medium with chemistry).
- non-traditional materials (e.g. chocolate).

It is noteworthy that most of these problems involve several nonlinear effects. HKS's approach has been to design a modular code: "modular" in that sense that the libraries of elements, materials models, and analysis procedures are genuinely independent: any combinations can be used together in a model. This is difficult to achieve, but essential, since our experience has been that almost any guess that a particular combination is not likely to be used is wrong!

Quality Assurance

The combinatorial issue gives rise to a difficult Quality Assurance problem—we cannot test all combinations. HKS uses a "classical" QA approach (as defined by ISO 9000, or the NQA-1 standard for nuclear power plant design in the USA). The test suite for ABAQUS/Standard contains about 5000 cases. These are run at each release (they take 2–5 cpu days on a typical workstation). Regression tests are run nightly on the development versions, to help ensure that errors are not introduced in existing features by the addition of new ones. All development projects and bug fixes are documented, and are cross-referenced to code changes. Thus, the reason for each line of code is known and documented. We also conduct failure analysis: all bugs discovered after a release are diagnosed for causes. If it is deemed necessary, tests are developed to avoid similar failures in future.

ABAQUS/Standard has a strong reputation for quality but, like any complex software system, it contains bugs. Those that are known are documented, and this information is made available to users. Few of them are of major significance, and there is little demand for maintenance releases to correct errors. In this sense the extensive testing of the software prior to release, while expensive, appears to be effective.

Sources of Nonlinearity

In this section we discuss sources of nonlinearity in solid and structural problems. The purpose of the discussion is to provide some judgment about the level of difficulty associated with various nonlinear effects.

Geometric/kinematic nonlinearity

The simplest form of kinematic nonlinearity is stress stiffening—the violin string effect. This generally manifests itself in frequency shifts in vibration studies.

The opposite effect is buckling. The typical design case usually involves small displacements: the failure of a "stiff" shell-type structure to carry load in a membrane mode. Buckling introduces some tricky issues: the failure mode cannot easily be anticipated but the mesh must represent it, and structures are often imperfection sensitive. The post-buckling response may be unstable, and may even snap back. Thus, eigenvalue analysis alone is rarely sufficient: we also need to compute the load–displacement response.

Follower force issues can add complexity. The thin cylinder under external pressure is a simple illustration. The follower effect of the pressure loading (the loading changes direction as the cylinder collapses) reduces the collapse load by 30%. A more subtle case is the thin cantilever under an end load (Figure 1).

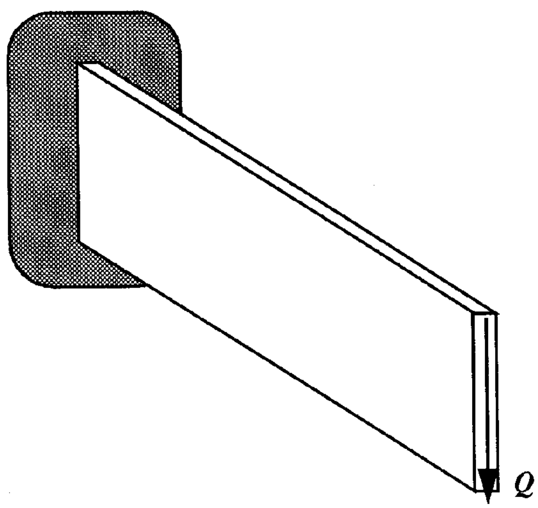

Figure 1. Cantilever under end load.

If Q is a "dead" load this is a classical static buckling case. But if Q follows the direction of the end of the cantilever (like a jet engine on the wing of an airplane), there is no static buckling load: the instability is a dynamic flutter.

Buckling problems provide plenty of pitfalls, and a long "learning curve" for beginners. Further, it is difficult to build help into a code. Nevertheless, buckling and collapse is a common, and successful, area of routine application of commercial codes by experienced analysts.

Large rotation, small strain

Many problems involve large rotations but only small strains. Examples include the roofs of convertibles, vehicle suspension systems, aircraft landing gear, and offshore pipelaying. A common issue in such cases is that structures are often very slender (in offshore pipelaying it is not uncommon to encounter pipes that are kilometers long but only a few inches in diameter). The usual displacement formulation is not suitable for such cases, because the bending and axial stiffnesses of the members are so very different. Instead, we use a mixed formulation. For example, the usual beam elements have displacement, **u**, and rotation, ϕ as degrees of freedom. To handle such slender problems we add N, the axial force in the element, as a variable (we also add the transverse shear forces, T_α, $\alpha = 1, 2$, in shear flexible beams). The appropriate compatibility conditions are added to the mechanical equilibrium equations to provide an adequate set of equations. This means that, even before any member buckles, we no longer have a positive definite "stiffness matrix."

A suitable rotation measure is needed to provide a convenient approach to handling large rotations. In this context "convenient" means that it should be easy for the user to impose prescribed motion. This is achieved by interpreting prescribed angular motion as prescribed angular velocity: $\Delta\phi$ (vector) prescribed over Δt is interpreted as a uniform angular velocity of $\Delta\phi/\Delta t$ during Δt. This is easy to understand, especially when prescribing compound motions ($\Delta\phi_1$ about the axis $\mathbf{p}_1$ followed by $\Delta\phi_2$ about $\mathbf{p}_2$, ...). However, the implication is that

$$\phi \neq \int \sum (\Delta\phi/\Delta t)\, dt$$

where ϕ is the total rotation, defined as $\phi = |\phi|$ radians about axis $\mathbf{p} = \phi/|\phi|$.

We use quaternions internally for computational efficiency, since this provides the quaternion product for compound rotations, and such products involve only scalar and vector operations with no singularities, regardless of the extent of the rotation.

Large strain

Large strains in solids arise from soft (visco-)elastic response, ductility (crystalline plasticity) and frictional flow (granular materials). Except for voided materials the deformation is predominantly deviatoric: there is hardly any change in the volume of the material. Thus, effective analysis requires models that operate correctly with severe constraint on their volume change. Usually this is not a problem in solid mechanics: some (but not all) elements in common use generally do an adequate job for the rate form of incompressible deformation encountered in plastic flow models, while mixed formulations are available for fully incompressible cases like elastomers.

ABAQUS uses a Lagrangian formulation for solids because material behavior depends on total deformation or is history dependent. However, this means that large strains (> 50% or so) lead to mesh distortion. Rezoning then becomes necessary. Rezoning is essential for some bulk metal forming and for some elastomer applications. In sheet metal forming the strains are usually limited to less than 30% by formability limits. Nevertheless rezoning is still desirable, to model significant local details, such as tight curvatures, wrinkling and necking. Since the ABAQUS-based subset offered in MSC/ARIES does not provide rezoning, its application to large strain problems is limited to those for which rezoning is not needed. In spite of this limitation, a wide range of large strain problems can still be modeled with suitable meshing.

Material nonlinearity

Solid materials offer a very broad range of behaviors that are often complex. Some common cases are as follows:

Elasticity

Isothermal mechanical characterization of elastic materials is simple in principle: the behavior is defined by a strain energy density potential

$$W(\varepsilon)$$

where W is the strain energy density and ε is the (large) strain. Since most common elastomers are isotropic this can be simplified to

$$W(\lambda_1,\lambda_2,\lambda_3),$$

where λ_i are the principal stretches. Further, since the material is usually incompressible (except for foamed materials: see below),

$$\lambda_1\lambda_2\lambda_3 \approx 1,$$

so that

$$W = W(\lambda_1,\lambda_2).$$

Thus, we need a suitable smooth function of two variables. Ogden's model is a useful choice:

$$W = 2\sum_i \frac{\mu_i}{\alpha_i^2}\left[\tilde{\lambda}_1^{\alpha_i} + \tilde{\lambda}_2^{\alpha_i} + \tilde{\lambda}_3^{\alpha_i} - 3\right],$$

where the $\tilde{\lambda}_i$ are the deviatoric parts of the stretch:

$$\tilde{\lambda}_i = J^{-\frac{1}{3}}\lambda_i,$$

and μ_i, α_i, $i = 1, 2, \ldots$ are material parameters.

Ogden's model is useful because it is easy to calibrate (the response of many common elastomers can be fit accurately to quite large strain with two terms) and its stability is easily proved. Since the material is (almost) incompressible, mixed formulation elements are needed except for plane stress calculations. ABAQUS/Standard uses quads and bricks with constant or locally linear pressure stress, with very few difficulties.

Foamed elastomers (such as seat cushions) can be modeled with a modified Ogden form:

$$W = 2\sum_i \frac{\mu_i}{\alpha_i^2}\left[\lambda_1^{\alpha_i} + \lambda_1^{\alpha_i} + \lambda_1^{\alpha_i} - 3 + \frac{1}{\beta_i}\left[J^{-\alpha_i\beta_i} - 1\right]\right],$$

where μ_i, α_i, β_i, $i = 1, 2, \ldots$ are material parameters.

Viscoelastic modeling is expressed as a complex, frequency dependent modulus in small vibration cases, and as a Prony series for time domain analysis. While some important materials exhibit viscoelastic behavior, filled rubbers exhibit internal friction, not viscosity, and viscoelastic modeling is only a coarse approximation to such behavior.

Analysis of elastic and viscoelastic components at large strain usually also involves contact, since rubber-like materials are often used as seals or gaskets. Adjacent components are usually so stiff compared to the elastomer that they can be considered rigid in comparison. Contact algorithms for a deforming body interacting with a rigid body generally work quite well. Thus, many design problems with elastomers are routine and reliable.

Ductile materials

Ductility is associated with inelastic—"plastic"—deformation of crystalline materials. Macroscopic models of ductile behavior are generally built on the same, fundamentally simple, foundation, using four concepts:

A strain decomposition:

$$\mathbf{F} = \mathbf{F}^{el} \cdot \mathbf{F}^{pl},$$

where $\mathbf{F}$ is the deformation gradient.

A stress limit for purely elastic deformation (the "yield surface"):

$$G(\sigma, \theta, h^{\alpha}) \leq 0$$

where σ is the stress, θ is the temperature, and h^{α} are hardening variables.

A flow rule:

$$\dot{\varepsilon}^{pl} = \frac{\partial H}{\partial \sigma},$$

where $H(\sigma, \theta, h^{\alpha})$ is the flow potential.

A hardening model, giving the evolution of the h^{α}.

Fortunately, relatively simple mathematical forms for G and H are quite suitable for common applications. For example, Mises yield with associated plastic flow ($H = G$) is accurate for gross flow of metals at low temperature. And "Coulomb" yield with purely deviatoric (and, therefore, non-associated) flow is useful for frictional materials such as soils (although in this case we actually use a smoothed form of the Coulomb yield surface).

The most insecure part of theory is often the hardening model. Here, again, simple models work well for important applications. Perfect plasticity ($h^{\alpha} = 0$) and isotropic hardening,

$$h^{1} = h^{1}(e^{pl}),$$

where e^{pl} is the scalar equivalent plastic strain, work well for many problems involving gross plastic flow or relatively monotonic straining. Sophisticated hardening models are needed for cyclic loading (to model the "Baushinger effect"), especially at high temperatures where rate dependence also becomes important.

Since the flow rule is defined in rate form and the hardening evolution laws also generally have a rate form, time integration is required in the numerical implementation of plasticity models. Stable, accurate, efficient integration methods are now well known. Usually the backward Euler method is used in implicit codes like ABAQUS/Standard.

In summary, relatively simple models are available to simulate the behavior of many common ductile materials, and they generally work well for routine applications. Low cycle fatigue problems are often the most challenging, because of the need for more sophisticated hardening models that are commonly used and, often, the presence of high temperatures, requiring visco-plastic (creep) modeling combined with yield. Appropriate models for these applications are more complex than those built into ABAQUS/Standard.

Issues also arise in the context of microscale (but still continuum) metal plasticity models, where the material appears to exhibit a characteristic length scale. One possible approach is to use non-local constitutive modeling. However, this introduces data storage issues in a finite element code, as well as complexities (and non-symmetry) in Jacobian calculations in implicit codes. Alternatively, higher order continuum theories involving strain gradients and couple stresses are used. This implies the need for C^1 continuous finite elements in three dimensions: a relatively unexplored area. Metal plasticity at the microscale is still a research topic.

Brittle behavior

Concrete, rock and ceramics all exhibit brittle behavior. The analysis problem is that this causes localization: deformation and failure concentrates into a "fracture process zone" whose dimension is typically small compared to physical (finite element) size. Unless some accommodation is made, results are totally mesh dependent. Consider a brittle member in tension. If the softening behavior of the material is defined as stress–strain behavior, the overall load–displacement response is entirely dependent on the size of the smallest element (Figure 2).

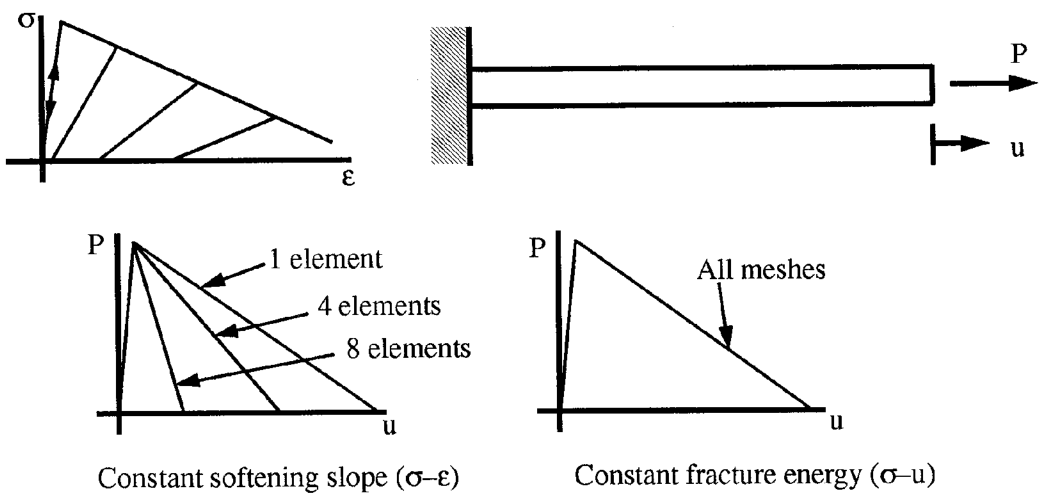

Figure 2. Modeling a brittle member under tension.

The design approach is to recognize this fracture energy as a material property and thus recast the softening response as stress–displacement behavior. This, together with characteristic element size measures in the finite element model, allow failure definition, at least to the size of the mesh discretization.

Brittle material modeling often involves other issues, such as rebar/concrete interaction. Further, the instability associated with localization can cause severe numerical difficulties in both implicit and explicit codes. Design analysis involving brittle material failure is still, therefore, an area for specialists.

Contact

Contact without friction is a purely kinematic constraint. If this is the only nonlinearity in the problem, it is usually straightforward to analyze. Algorithms must be appropriate: our experience suggests that we should discard penalty methods and only use Lagrange multipliers or elimination.

There are some subtleties. For example, do we have one, two, or three constraints at a point? The discretized finite element model itself does not know whether it is smooth or faceted—geometric information must be associated with the model to provide the answer (Figure 3).

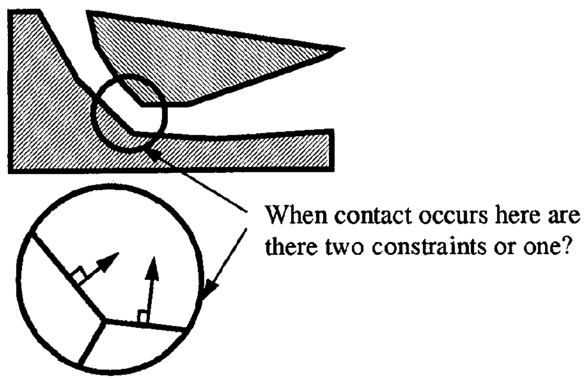

Figure 3. Contact constraints.

Friction is a complex surface nonlinearity. Coulomb friction is a non-associated flow plasticity model applied to surface interaction. With smooth surfaces the elastic response is essentially rigid. However, this is one case where our experience has been that a penalty method (with adaptive penalty selection) is the only generally satisfactory approach. We generally use an allowable (non-zero) elastic shear slip, chosen automatically, to define the penalty (Figure 4).

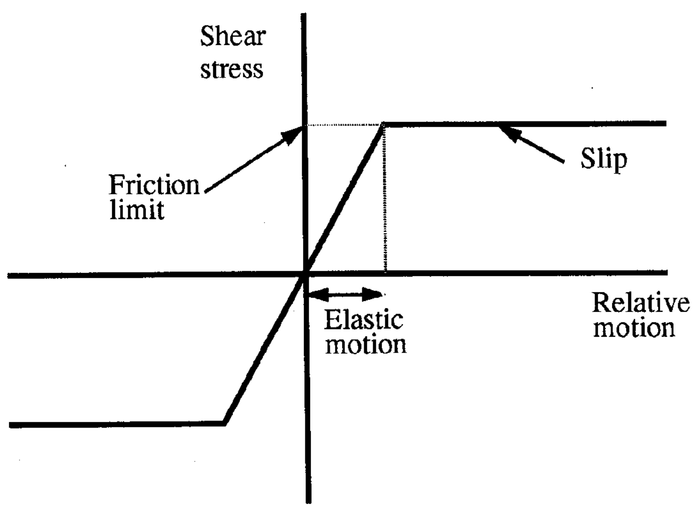

Figure 4. Simple frictional model.

This Coulomb model, together with a shear stress cut-off, is the most commonly used frictional model. More general friction laws are beyond the scope of this discussion.

Solution algorithms

The general problem is mechanical equilibrium:

$$M^{NM}\ddot{u}^{M} + I^{N} - P^{N} = 0 ,$$

where M^{NM} is the mass matrix, $\ddot{u}^{N}$ are the nodal accelerations,

$$I^{N}(u^{M}) = \int_{V} \beta^{N} \cdot \sigma \, dV$$

are the internal forces; $\beta^{N}(\mathbf{x})$ is the strain rate–displacement rate transformation; $\boldsymbol{\sigma}$ is the Cauchy stress; V is the current volume;

$$P^{N}(u^{M}) = \int_{S} \mathbf{N}^{N} \cdot \mathbf{p} \, dS + \int_{V} \mathbf{N}^{N} \cdot \mathbf{r} \, dV$$

are the external forces; $\mathbf{N}^{N}(\mathbf{x})$ are the interpolation functions, $\mathbf{p}$ are the surface tractions, and $\mathbf{r}$ are the volumetric loads.

The d'Alembert forces, $M^{NM}\ddot{u}^{M}$, are negligible in static cases.

Often we also need to include general constraints of the form $H^{P}(u^{N}) = 0$, such as incompressibility:

$$\int_{V} L^{P}(J - J^{th}) \, dV = 0 ,$$

where $J = \det(\mathbf{F})$ is the total volume change at a point, and J^{th} is the volumetric thermal expansion. $L^{P}(\mathbf{x})$ is an assumed pressure stress variation, defining the pressure stress, p, to be interpolated as $p = L^{P}\bar{p}^{P}$, where $\bar{p}^{P}$ are constraint variables.

In ABAQUS/Standard we use constant pressure in first order elements, locally linear pressure in second order elements for this purpose (Figure 5).

p = constant

$p = a_0 + a_1 x + a_2 y + a_3 z$

Figure 5. Mixed formulation elements for incompressible cases.

Although some of the resulting elements do not fully satisfy mathematical stability requirements (to avoid "checkerboarding"), in practice they generally work well except in highly confined cases subject to high pressure.

General approach to solution

Explicit time integration of the dynamic system is the most robust method for severely discontinuous nonlinearity (complex contact problems). The integration in time is

$$\ddot{u}\Big|_t = \left[\dot{u}\Big|_{t+\frac{1}{2}\Delta t} - \dot{u}\Big|_{t-\frac{1}{2}\Delta t}\right] / \Delta t$$

with the same formula used for velocities so that, with fixed Δt,

$$\Delta u\Big|_{t+\Delta t} = \Delta u\Big|_t + \ddot{u}\Big|_t \Delta t^2 .$$

The algorithm is only conditionally stable: for an undamped system,

$$\Delta t \le \frac{2}{\omega_{max}} = (\text{min. period})/\pi$$

or, stated another way, information can propagate no further than to the nearest neighbor nodes in one time increment.

Consider a structural calculation to which the method is typically applied: a front end car crash. The members are mostly made from steel, so that the wave speed is about 5000 m/sec. The smallest element size will typically be about 30mm, so that

$$\Delta t = \frac{30}{5} \times 10^{-6} = 6\ \mu\text{sec}.$$

The event duration is about 0.4 sec, so that about 70,000 increments are necessary. Such calculations are therefore typically done on vector processors.

The advantages of this approach is that it is robust (in the sense that it always computes numbers), the cost only rises linearly with model size (so that larger models are tractable) and the method involves purely vector operations, offering possibilities for a high level of vectorization, and parallel processing (although complexities like contact make this difficult).

The method has several disadvantages. It is computationally intensive, and is therefore only effective with specially designed codes (hence we see it offered in MSC/DYTRAN or ABAQUS/Explicit, but not—yet—in more general codes like MSC/NASTRAN or ABAQUS/Standard: the overhead of such a code is too costly to provide a competitive package). The stability limit makes the method inefficient for quasi-static cases ("dynamic relaxation" takes many iterations). All degrees of freedom must have inertia—we cannot have any massless nodes. Time step stability is tricky in unstable cases, such as strain softening. It is costly to include constraints: since we do not want to solve equations, we cannot use Lagrange multipliers, while realistic penalties can introduce unacceptably short time steps because of stability issues. It is also undesirable to use higher order elements, since we cannot find a suitable lumped mass for such elements.

Overall, the method is highly valuable for certain very nonlinear cases, and has become the standard method for several important problems. These include some highly discontinuous quasi-static cases, such as sheet metal forming problems, simply because of the method's reliability in resolving cases that involve complex contact.

Implicit methods require that we solve nonlinear equations. For this purpose we generally use Newton's method:

$$\begin{bmatrix} \partial R^N / \partial u^M & \partial R^N / \partial \bar{p}^Q \\ \partial H^P / du^M & \partial H^P / \partial \bar{p}^Q \end{bmatrix} \begin{bmatrix} c_u^M \\ c_p^Q \end{bmatrix} = \begin{bmatrix} R^N \\ -H^P \end{bmatrix},$$

$$u^N = u^N + c_u^N,$$
$$\bar{p}^P = \bar{p}^P + c_p^P,$$

iterate,

where

$$R^N = P^N - I^N - M^{NM} \ddot{u}^M$$

is the residual in the equilibrium equations and

$$\begin{bmatrix} \partial R^N / \partial u^M & \partial R^N / \partial \bar{p}^Q \\ \partial H^P / du^M & \partial H^P / \partial \bar{p}^Q \end{bmatrix}$$

is the "Jacobian" (the stiffness matrix) of the system.

We extend the method with such techniques as the arc length constraint approach, line search, automatic increment size selector, etc.

The advantages of this approach is that it works well for broad range of problems. It also allows us to include general constraints via Lagrange multipliers, as indicated above.

It has several disadvantages. To achieve quadratic convergence it requires an "exact" Jacobian. Often this is difficult to find, although symbolic manipulation packages have helped here. The Jacobian is often not a symmetric matrix (friction, or any other non-associated plastic flow model, provides a non-symmetric Jacobian). The solution cost rises with the cube of the model size, because we must solve a linear system, and we generally do so with a direct (Gauss elimination) approach, since iterative methods are not reliable for the range of applications we need to address. Finally, the method may fail in severe cases. Nevertheless, it works well enough that this is the standard approach offered in ABAQUS/Standard.

Given that both methods have limitations, we might consider combining them—switching between methods as necessary. This looks attractive for certain important applications. For example, in sheet forming we can use the explicit method to handle the complex contact problem during the punching phase, then transfer to implicit analysis to do springback efficiently, as a static analysis. Having thus obtained the formed component we can continue in the implicit code to study buckling, vibration, thermal stress, etc. on the component with the actual thickness variation and residual (self-equilibrating) stresses, or incorporate the component into a larger model for a crash study, done again with the explicit method.

This combined approach inherits some of the limitations of both methods. Applications are restricted to cases where explicit modeling can be used (for example, the model cannot include any general constraints, higher order elements or massless nodes): the "richness" of modeling options in typical general purpose codes is not available. In the sheet forming application the implicit code needs to use a direct equation solver (at least at this stage of development: Belytschko recently noted that he has yet to see an iterative solver that can handle general, thin shell problems robustly). The problem size is thus limited to what can be handled by such a solver. This may be considerably less than the explicit code can manage, although for the sheet forming example mentioned above this is not likely to be a serious limitation with currently available computer systems.

Closure

Some quite complex nonlinear simulations in solid and structural mechanics can now be used in design. The performance and cost of modern computers makes computationally intensive calculations possible, suitable models (such as constitutive models) are available for common materials, and algorithms are sufficiently robust to treat many important cases.

Many issues remain open. We lack good models for brittle materials, composites outside the linear response range, metals at high temperatures or strain rates, flow localization in metals (sheet formability limits) and microscale ductility studies. We only have simple models for surface mechanical interaction ("friction"). We need more robust and efficient solution algorithms, especially for cases involving bifurcation, instability, discontinuous nonlinearity (contact) and combinations of these effects.

Currently we offer *finite element solvers*: codes that compute numbers on a given mesh. Adaptive methods are becoming available for linear analysis. These offer the promise of providing *boundary value problem solvers*—software that can compute an approximate solution of known (or adequately bounded) accuracy to a boundary value problem. Error estimators and adaptive methods are likely to be fundamentally different for nonlinear problems, where p-based adaptivity is only of limited value because of the inherent lack of smoothness in many cases. Instead, nonlinear analysis is likely to continue to use low order elements and h-adaptivity.

Adaptivity requires automatic meshing. Currently there are no automatic meshing algorithms that provide hexahedral elements for general geometries. Some interesting approaches are under development, such as the Sandia "plastering" algorithm, and Cecil Armstrong's "medial surface" algorithm. Automatic meshing with tetrahedra elements is available, but not entirely robust. Further, the first order tetrahedron is a very poor element (it exhibits extremely slow convergence with respect to mesh size). The second order tetrahedron is expensive. It lacks a satisfactory lumped mass, which would be needed for practical use of explicit integration, and it does not provide satisfactory distributed of uniform surface loads, as is needed for contact. These problems might be overcome. Alternatively, we might avoid the problem of meshing altogether, by using meshless analysis (Belytschko, Lu and Gu, 1994).

These are future prospects. Today MSC offers a nonlinear finite element solver, based on ABAQUS/Standard, integrated into MSC/ARIES. The nonlinear capabilities of this product are mature and proven. They cannot model all nonlinear problems. But they offer the careful analyst the opportunity to study a wide range of important applications that include many of the complex combinations of nonlinear effects that arise in practice.