CLOSED-FORM STATIC ANALYSIS WITH INERTIA RELIEF AND DISPLACEMENT-DEPENDENT LOADS USING A MSC/NASTRAN DMAP ALTER

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Abstract

Solving for the displacements of free-free coupled systems acted upon by static loads is commonly performed throughout the aerospace industry. Many times, these problems are solved using static analysis with inertia relief. This solution technique allows for a free-free static analysis by balancing the applied loads with inertia loads generated by the applied loads. For some engineering applications, the displacements of the free-free coupled system induce additional static loads. Hence, the applied loads are equal to the original loads plus displacement-dependent loads. Solving for the final displacements of such systems is commonly performed using iterative solution techniques. Unfortunately, these techniques can be time-consuming and labor-intensive. Since the coupled system equations for free-free systems with displacement-dependent loads can be written in closed-form, it is advantageous to solve for the displacements in this manner. Implementing closed-form equations in static analysis with inertia relief is analogous to implementing transfer functions in dynamic analysis. Using a MSC/NASTRAN DMAP Alter, displacement-dependent loads have been included in static analysis with inertia relief. Such an Alter has been used successfully to efficiently solve a common aerospace problem typically solved using an iterative technique.

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Nomenclature

Abbreviations

DOF	Degree-of-freedom
DMAP	Direct Matrix Abstraction Program
ELV	Expendable Launch Vehicle

Matrices

F	Applied forces
I	Identity
K	Stiffness
M	Mass
P	Applied loads
T	Displacements transformation
u	Displacements
$\ddot{u}$	Accelerations
η	Steady-state accelerations
μ	Steady-state accelerations transformation
ρ	Loads transformation
Φ	Mode shapes

Set Notation

a	a-set (assembled DOF)
g	g-set (structural grid DOF)
l	l-set (left over DOF)
r	r-set (reference DOF)
u_1	u_1-set (subset of l-set DOF)
y	y-set (subset of l-set DOF)
σ	σ-set (r-set + y-set DOF)

Introduction

Solving for the displacements of free-free coupled systems acted upon by static loads is commonly performed throughout the aerospace industry. Such analyses are performed for ELV/spacecraft systems during assumed-static, or quasistatic, phases of flight. For these flight event analyses, it is assumed that only steady-state loads act on the system and system transient responses have dampened out. Many times, these problems are solved using static analysis with inertia relief. This solution technique allows for a free-free static analysis by balancing the applied loads with inertia loads. Static analysis with inertia relief is offered in MSC/NASTRAN via Solution 91 [1].

For some engineering applications, the displacements of a free-free coupled system induce additional static loads. Hence, the applied loads are equal to the original loads plus displacement-dependent loads. Such is the case when analyzing ELV/spacecraft systems acted upon by quasistatic aerodynamic loads [2]. Quasistatic aerodynamic loads are generated as the system flies through the atmosphere at an angle-of-attack. The system static deformations cause local changes in the angle-of-attack which result in additional aerodynamic loads. The system will reach a state of static equilibrium under the static-aeroelastic loading. Solving for the final displacements of such systems is commonly performed using iterative solution techniques. Unfortunately, these techniques can be time-consuming and labor-intensive.

Since the free-free coupled system equations with displacement-dependent loads can be written in closed-form, it is advantageous to solve for the final displacements in this manner. The objective of this work was to develop a closed-form methodology for including displacement-dependent loads during static analysis with inertia relief using MSC/NASTRAN. Implementing closed-form equations in static analysis with inertia relief is analogous to implementing transfer functions in dynamic analysis in that induced load terms are added to the system stiffness resulting in a nonsymmetric matrix. A MSC/NASTRAN DMAP Alter has been developed for including displacement-dependent loads during static analysis with inertia relief. The Alter has been used successfully to efficiently solve a quasistatic ELV/spacecraft aerodynamic loads problem once solved using an iterative solution technique.

Closed-form static analysis with displacement-dependent loads is illustrated in the next section. A simple example problem is solved to demonstrate the basic principles behind the development of the new Alter. In a subsequent section, the underlying theory of closed-form static analysis with inertia relief and displacement-dependent loads is described. Implementation of the theory within a MSC/NASTRAN DMAP Alter is then explained. Lastly, a quasistatic ELV/spacecraft aerodynamic loads problem is solved to demonstrate the accuracy of using the new Alter versus using an iterative solution technique.

Closed-form Static Analysis with Displacement-dependent Loads

As previously stated, the objective of this work was to develop a methodology using MSC/NASTRAN, whereby, a static analysis with displacement-dependent loads, typically solved using iterative solution techniques, could be solved in closed-form. To demonstrate the basic principles of closed-form static analysis with displacement-dependent loads, consider the three DOF system with applied forces shown in Figure 1. Applied forces f_1 and f_2 are assumed constant. Applied force f_3 is assumed to be a function of displacements u_1 and u_2, or

$$f_3 = c(u_1 - u_2) \tag{1}$$

where c is a constant. The static equations describing the system are

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \tag{2}$$

or

$$[K]\{u\} = \{F\} \tag{3}$$

When not using a closed-form solution technique, the solution for the displacements $\{u\}$ must be found using an iterative solution technique because the applied force f_3 shown in Eq. (2) is not known. The system static equations rewritten for an iterative solution are

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix} \begin{Bmatrix} u_1^{i+1} \\ u_2^{i+1} \\ u_3^{i+1} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3^{\,i} \end{Bmatrix} \tag{4}$$

or

$$[K]\{u^{i+1}\} = \{F^{\,i}\} \tag{5}$$

where i signifies the iteration number (i = 0, 1, 2, ...). To begin solving for $\{u^{i+1}\}$ using an iterative solution technique, an initial value for applied force f_3 is assumed. Let f_3 be equal to zero when i=0; hence,

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix} \begin{Bmatrix} u_1^{1} \\ u_2^{1} \\ u_3^{1} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ 0 \end{Bmatrix} \tag{6}$$

A first set of system displacements $\{u^1\}$ is solved for using Eq. (6). Once $\{u^1\}$ is solved for, the applied force f_3^1 is calculated, the applied forces $\{F^i\}$ are updated, and system displacements $\{u^2\}$ are solved for. This procedure continues until the difference in system displacements between two successive iterations satisfies a convergence criterion, or

$$\|\,\{u^{i+1}\} - \{u^{\,i}\}\,\| \le \varepsilon \qquad (\text{where } \varepsilon << 1) \tag{7}$$

At this point, the solution has converged, and the system displacements are known.

The displacements for the system shown in Figure 1 can also be solved for in closed-form. Let a fourth DOF, u_4, be defined as the difference between system displacements u_1 and u_2, or

$$u_4 = u_1 - u_2 \tag{8}$$

The displacement-dependent applied force f_3 can then be written as

$$f_3 = cu_4 \tag{9}$$

If the definition of u_4 is included within the system stiffness matrix [K] and the variable applied force f_3 is moved to the left-hand-side of the system static equations, the system equations become

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -c \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{Bmatrix} \tag{10}$$

Adding the fourth linear equation to the system and moving the variable applied force term to the left-hand-side is the procedure for rewriting the iterative solution shown by Eq. (4) as a closed-form solution. Equation (10) defines a closed-form static problem with displacement-dependent loads. The system displacements {u} can be solved for immediately.

Before describing the general implementation of closed-form static analysis with inertia relief and displacement-dependent loads within MSC/NASTRAN Solution 91, it is beneficial to look at the example system stiffness matrix shown in Eq. (10). Note that the matrix is nonsymmetric. Adding the fourth DOF to the problem is analogous to adding extra points in MSC/NASTRAN dynamic analysis in that the once symmetric matrices become nonsymmetric. Partitioning [K] of Eq. (10) into original physical DOF and added DOF partitions,

$$[K] = \begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} \tag{11}$$

where

$$[K_{11}] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix} \tag{12}$$

$$[K_{12}] = \begin{bmatrix} 0 \\ 0 \\ -c \end{bmatrix} \tag{13}$$

$$[K_{21}] = \lfloor -1 \quad 1 \quad 0 \rfloor \tag{14}$$

and

$$[K_{22}] = \lfloor 1 \rfloor \tag{15}$$

The physical stiffness partition, $[K_{11}]$, is equal to the original system stiffness matrix shown in Eq. (2). The two lower partitions, $[K_{21}]$ and $[K_{22}]$, contain elements of the linear equation defining the added DOF displacement shown by Eq. (8). The upper-right partition, $[K_{12}]$, defines the variable force applied to the physical DOF due to a unit displacement of the added DOF shown by Eq. (9).

In summary, static equations with displacement-dependent loads solved using an iterative solution technique can be solved in closed-form by:

1. Generating additional DOF and writing linear equations defining their displacements as functions of displacements of the physical DOF and including these equations within the system static equations.

2. Redefining the displacement-dependent load relationships as functions of the additional DOF displacements and including these relationships within the system static equations.

These basic principles were used when developing a MSC/NASTRAN DMAP Alter to Solution 91 for performing closed-form static analysis with inertia relief and displacement-dependent loads.

Closed-form Static Analysis with Inertia Relief and Displacement-dependent Loads

Including displacement-dependent loads during static analysis with inertia relief was implemented within MSC/NASTRAN Solution 91 using a DMAP Alter. The underlying theory upon which the Alter is based stems from the basic concepts presented in the previous section. Special considerations were made for including inertia relief effects during development of the Alter. The goal of the development effort was to generate closed-form static equations of the form shown by Eq. (10) to efficiently perform static analyses with inertia relief and displacement-dependent loads.

Before developing closed-form static equations with inertia relief and displacement-dependent loads, it is beneficial to review the basic theory of static analysis with inertia relief. The equations-of-motion for all DOF of a free-free coupled system under a steady-state loading condition are

$$[M_{gg}]\{\ddot{u}_g\} + [K_{gg}]\{u_g\} = \{\bar{P}_g\} \tag{16}$$

After accounting for DOF defined via multi-point and single-point constraints, the system equations are reduced from g-set size to a-set size [1]; hence,

$$[M_{aa}]\{\ddot{u}_a\} + [K_{aa}]\{u_a\} = \{\bar{P}_a\} \tag{17}$$

To solve for the displacements of the free-free system represented by Eq. (17), an inertia relief solution technique can be used because the original system stiffness matrix is singular [3]. The a-set DOF are the union of statically-determinate reference DOF (r-set) and the complement of the r-set DOF (l-set). Writing Eq. (17) in partitioned form,

$$\begin{bmatrix} [\bar{M}_{rr}] & [M_{rl}] \\ [M_{lr}] & [M_{ll}] \end{bmatrix} \begin{Bmatrix} \{\ddot{u}_r\} \\ \{\ddot{u}_l\} \end{Bmatrix} + \begin{bmatrix} [K_{rr}] & [K_{rl}] \\ [K_{lr}] & [K_{ll}] \end{bmatrix} \begin{Bmatrix} \{u_r\} \\ \{u_l\} \end{Bmatrix} = \begin{Bmatrix} \{\bar{P}_r\} \\ \{\bar{P}_l\} \end{Bmatrix} \tag{18}$$

Under the steady-state loading condition, the system deforms elastically and accelerates as a rigid-body. Using a rigid-body transformation [4], the a-set DOF steady-state accelerations are written as

$$\begin{Bmatrix} \{\ddot{u}_r\} \\ \{\ddot{u}_l\} \end{Bmatrix} = \begin{bmatrix} [I_{rr}] \\ -[K_{ll}]^{-1}[K_{lr}] \end{bmatrix} \{\ddot{u}_r\} = [\Phi_{ar}]\{\ddot{u}_r\} \tag{19}$$

Using Eq. (19), the system equations shown by Eq. (18) are rewritten as

$$\begin{bmatrix} [K_{rr}] & [K_{rl}] \\ [K_{lr}] & [K_{ll}] \end{bmatrix} \begin{Bmatrix} \{u_r\} \\ \{u_l\} \end{Bmatrix} = \begin{Bmatrix} \{\bar{P}_r\} \\ \{\bar{P}_l\} \end{Bmatrix} - \begin{bmatrix} [\bar{M}_{rr}] & [M_{rl}] \\ [M_{lr}] & [M_{ll}] \end{bmatrix} \begin{bmatrix} [I_{rr}] \\ -[K_{ll}]^{-1}[K_{lr}] \end{bmatrix} \{\ddot{u}_r\} \tag{20}$$

Premultiplying Eq. (20) by $[\Phi_{ar}]^T$,

$$\begin{aligned} \{0_r\} &= [\Phi_{ar}]^T\{\bar{P}_a\} - [\Phi_{ar}]^T[M_{aa}][\Phi_{ar}]\{\ddot{u}_r\} \\ &= [\Phi_{ar}]^T\{\bar{P}_a\} - [M_{rr}]\{\ddot{u}_r\} \end{aligned} \tag{21}$$

Solving for the r-set DOF steady-state accelerations $\{\ddot{u}_r\}$ from Eq. (21),

$$\{\ddot{u}_r\} = [M_{rr}]^{-1}[\Phi_{ar}]^T\{\bar{P}_a\} \tag{22}$$

Because the free-free system has rigid-body modes, displacements $\{u_r\}$ can be set arbitrarily. Letting $\{u_r\}=\{0_r\}$, the lower partition of Eq. (20) becomes

$$[K_{ll}]\{u_l\} = \{\bar{P}_l\} - [M_{la}][\Phi_{ar}]\{\ddot{u}_r\} \tag{23}$$

where $[M_{la}]$ is the lower partition of the mass matrix of Eq. (18). Defining the right-hand-side of Eq. (23) as $\{P_l\}$, Eq. (23) simplifies to

$$[K_{ll}]\{u_l\} = \{P_l\} \tag{24}$$

Equation (24) is the inertia relief solution for the l-set DOF displacements shown originally in Eq. (18).

Now let the static equations with inertia relief be expanded to include displacement-dependent loads. Beginning with the a-set DOF equations,

$$[M_{aa}]\{\ddot{u}_a\} + [K_{aa}]\{u_a\} = \{\bar{P}_a\} + [\bar{\rho}_{aa}]\{u_a\} \tag{25}$$

where $[\bar{\rho}_{aa}]$ is a matrix for transforming the a-set DOF displacements into loads acting on the system. Rewriting Eq. (25) according to r-set and l-set DOF partitions,

$$\begin{bmatrix} [\bar{M}_{rr}] & [M_{rl}] \\ [M_{lr}] & [M_{ll}] \end{bmatrix} \begin{Bmatrix} \{\ddot{u}_r\} \\ \{\ddot{u}_l\} \end{Bmatrix} + \begin{bmatrix} [K_{rr}] & [K_{rl}] \\ [K_{lr}] & [K_{ll}] \end{bmatrix} \begin{Bmatrix} \{u_r\} \\ \{u_l\} \end{Bmatrix} = \begin{Bmatrix} \{\bar{P}_r\} \\ \{\bar{P}_l\} \end{Bmatrix} + \begin{bmatrix} [\bar{\rho}_{rr}] & [\bar{\rho}_{rl}] \\ [\bar{\rho}_{lr}] & [\bar{\rho}_{ll}] \end{bmatrix} \begin{Bmatrix} \{u_r\} \\ \{u_l\} \end{Bmatrix} \tag{26}$$

To facilitate further development, let the l-set DOF be the union of user-defined DOF (u_1-set) and the complement of the u_1-set DOF (y-set). The u_1-set DOF have no mass or stiffness and are added to the system DOF for applying the displacement-dependent loads. These DOF are analogous to extra points used to define transfer functions in MSC/NASTRAN dynamic analysis. The y-set DOF are those of the original system model. Let the displacement-dependent loads be defined solely by the u_1-set DOF displacements, or

$$[\bar{\rho}_{aa}]\{u_a\} = [\bar{\rho}_{au_1}]\{u_{u_1}\} \tag{27}$$

This is accomplished by defining $\{u_{u_1}\}$ via a set of linear equations which describe the system displacement dependencies, or

$$\begin{bmatrix} [T_{u_1r}] & [T_{u_1y}] & [T_{u_1u_1}] \end{bmatrix} \begin{Bmatrix} \{u_r\} \\ \{u_y\} \\ \{u_{u_1}\} \end{Bmatrix} = \{0_{u_1}\} \tag{28}$$

Taking into account the y-set and u_1-set partitions of the l-set DOF, Eq. (26) is rewritten as

$$\begin{bmatrix} [\bar{M}_{rr}] & [M_{ry}] & [0_{ru_l}] \\ [M_{yr}] & [M_{yy}] & [0_{yu_l}] \\ [0_{u_lr}] & [0_{u_ly}] & [0_{u_lu_l}] \end{bmatrix} \begin{Bmatrix} \{\ddot{u}_r\} \\ \{\ddot{u}_y\} \\ \{0_{u_l}\} \end{Bmatrix} + \begin{bmatrix} [K_{rr}] & [K_{ry}] & [0_{ru_l}] \\ [K_{yr}] & [K_{yy}] & [0_{yu_l}] \\ [T_{u_lr}] & [T_{u_ly}] & [T_{u_lu_l}] \end{bmatrix} \begin{Bmatrix} \{u_r\} \\ \{u_y\} \\ \{u_{u_l}\} \end{Bmatrix} = \begin{Bmatrix} \{\bar{P}_r\} \\ \{\bar{P}_y\} \\ \{0_{u_l}\} \end{Bmatrix} + \begin{bmatrix} [0_{rr}] & [0_{ry}] & [\bar{\rho}_{ru_l}] \\ [0_{yr}] & [0_{yy}] & [\bar{\rho}_{yu_l}] \\ [0_{u_lr}] & [0_{u_ly}] & [0_{u_lu_l}] \end{bmatrix} \begin{Bmatrix} \{0_r\} \\ \{0_y\} \\ \{u_{u_l}\} \end{Bmatrix} \tag{29}$$

As before, the system deforms elastically and accelerates as a rigid-body under the steady-state loading condition. Using a rigid-body transformation [4], the a-set DOF steady-state accelerations are written as

$$\begin{Bmatrix} \{\ddot{u}_\sigma\} \\ \{0_{u_l}\} \end{Bmatrix} = \begin{Bmatrix} \{\ddot{u}_r\} \\ \{\ddot{u}_y\} \\ \{0_{u_l}\} \end{Bmatrix} = \begin{bmatrix} [I_{rr}] \\ -[K_{yy}]^{-1}[K_{yr}] \\ [0_{u_lr}] \end{bmatrix} \{\ddot{u}_r\} = \begin{bmatrix} [\Phi_{\sigma r}] \\ [0_{u_lr}] \end{bmatrix} \{\ddot{u}_r\} = [\Phi_{ar}]\{\ddot{u}_r\} \tag{30}$$

where the σ-set DOF are the union of the r-set and y-set DOF (complement of u_l-set in a-set).

Using Eq. (30), the system equations shown by Eq. (29) are rewritten as

$$\begin{bmatrix} [K_{rr}] & [K_{ry}] & [0_{ru_l}] \\ [K_{yr}] & [K_{yy}] & [0_{yu_l}] \\ [T_{u_lr}] & [T_{u_ly}] & [T_{u_lu_l}] \end{bmatrix} \begin{Bmatrix} \{u_r\} \\ \{u_y\} \\ \{u_{u_l}\} \end{Bmatrix} = \begin{Bmatrix} \{\bar{P}_r\} \\ \{\bar{P}_y\} \\ \{0_{u_l}\} \end{Bmatrix} + \begin{bmatrix} [\bar{\rho}_{ru_l}] \\ [\bar{\rho}_{yu_l}] \\ [0_{u_lu_l}] \end{bmatrix} \{u_{u_l}\} - \begin{bmatrix} [\bar{M}_{rr}] & [M_{ry}] & [0_{ru_l}] \\ [M_{yr}] & [M_{yy}] & [0_{yu_l}] \\ [0_{u_lr}] & [0_{u_ly}] & [0_{u_lu_l}] \end{bmatrix} \begin{bmatrix} [I_{rr}] \\ -[K_{yy}]^{-1}[K_{yr}] \\ [0_{u_lr}] \end{bmatrix} \{\ddot{u}_r\} \tag{31}$$

Premultiplying Eq. (31) by $[\Phi_{ar}]^T$,

$$\begin{aligned} \{0_r\} &= [\Phi_{ar}]^T\{\bar{P}_a\} + [\Phi_{ar}]^T[\bar{\rho}_{au_l}]\{u_{u_l}\} - [\Phi_{ar}]^T[M_{aa}][\Phi_{ar}]\{\ddot{u}_r\} \\ &= [\Phi_{ar}]^T\{\bar{P}_a\} + [\Phi_{ar}]^T[\bar{\rho}_{au_l}]\{u_{u_l}\} - [M_{rr}]\{\ddot{u}_r\} \end{aligned} \tag{32}$$

Solving for the r-set DOF steady-state accelerations $\{\ddot{u}_r\}$ from Eq. (32),

$$\begin{aligned} \{\ddot{u}_r\} &= [M_{rr}]^{-1}[\Phi_{ar}]^T\{\bar{P}_a\} + [M_{rr}]^{-1}[\Phi_{ar}]^T[\bar{\rho}_{au_l}]\{u_{u_l}\} \\ &= \{\eta_r\} + [\mu_{ru_l}]\{u_{u_l}\} \end{aligned} \tag{33}$$

In Eq. (33), $\{\eta_r\}$ are the steady-state accelerations due to the directly applied loads, and $[\mu_{ru_l}]$ is a matrix for transforming system displacements into additional steady-state accelerations.

Because the free-free system has rigid-body modes, $\{u_r\}$ can be set arbitrarily. Letting $\{u_r\}=\{0_r\}$, the lower partition of Eq. (31) becomes

$$\begin{bmatrix} [K_{yy}] & [0_{yu_l}] \\ [T_{u_l y}] & [T_{u_l u_l}] \end{bmatrix} \begin{Bmatrix} \{u_y\} \\ \{u_{u_l}\} \end{Bmatrix} = \begin{Bmatrix} \{\bar{P}_y\} \\ \{0_{u_l}\} \end{Bmatrix} + \begin{bmatrix} [\bar{\rho}_{yu_l}] \\ [0_{u_l u_l}] \end{bmatrix} \{u_{u_l}\} - \begin{bmatrix} [M_{y\sigma}] \\ [0_{u_l \sigma}] \end{bmatrix} [\Phi_{\sigma r}] \{\ddot{u}_r\} \tag{34}$$

Substituting the accelerations shown by Eq. (33) into Eq. (34), collecting like terms, and simplifying,

$$\begin{aligned} \begin{bmatrix} [K_{yy}] & [0_{yu_l}] \\ [T_{u_l y}] & [T_{u_l u_l}] \end{bmatrix} \begin{Bmatrix} \{u_y\} \\ \{u_{u_l}\} \end{Bmatrix} &= \left(\begin{Bmatrix} \{\bar{P}_y\} \\ \{0_{u_l}\} \end{Bmatrix} - \begin{bmatrix} [M_{y\sigma}] \\ [0_{u_l \sigma}] \end{bmatrix} [\Phi_{\sigma r}][\eta_r] \right) + \left(\begin{bmatrix} [\bar{\rho}_{yu_l}] \\ [0_{u_l u_l}] \end{bmatrix} - \begin{bmatrix} [M_{y\sigma}] \\ [0_{u_l \sigma}] \end{bmatrix} [\Phi_{\sigma r}][\mu_{ru_l}] \right) \{u_{u_l}\} \\ &= \begin{Bmatrix} \{P_y\} \\ \{0_{u_l}\} \end{Bmatrix} + \begin{bmatrix} [\rho_{yu_l}] \\ [0_{u_l u_l}] \end{bmatrix} \{u_{u_l}\} \end{aligned} \tag{35}$$

where

$$\{P_y\} = \{\bar{P}_y\} - [M_{y\sigma}][\Phi_{\sigma r}]\{\eta_r\} \tag{36}$$

$$[\rho_{yu_l}] = [\bar{\rho}_{yu_l}] - [M_{y\sigma}][\Phi_{\sigma r}][\mu_{ru_l}] \tag{37}$$

To eliminate the displacement-dependent loads on the right-hand-side of Eq. (35), Eq. (35) is first rewritten as

$$\begin{bmatrix} [K_{yy}] & [0_{yu_l}] \\ [T_{u_l y}] & [T_{u_l u_l}] \end{bmatrix} \begin{Bmatrix} \{u_y\} \\ \{u_{u_l}\} \end{Bmatrix} = \begin{Bmatrix} \{P_y\} \\ \{0_{u_l}\} \end{Bmatrix} + \begin{bmatrix} [0_{yy}] & [\rho_{yu_l}] \\ [0_{u_l y}] & [0_{u_l u_l}] \end{bmatrix} \begin{Bmatrix} \{u_y\} \\ \{u_{u_l}\} \end{Bmatrix} \tag{38}$$

The second term on the right-hand-side of Eq. (38) is then moved to the left-hand-side to give

$$\begin{bmatrix} [K_{yy}] & -[\rho_{yu_l}] \\ [T_{u_l y}] & [T_{u_l u_l}] \end{bmatrix} \begin{Bmatrix} \{u_y\} \\ \{u_{u_l}\} \end{Bmatrix} = \begin{Bmatrix} \{P_y\} \\ \{0_{u_l}\} \end{Bmatrix} \tag{39}$$

Equation (39) is the inertia relief solution for the l-set DOF displacements shown originally in Eq. (26).

The goal of the above development was to generate closed-form static equations of the form shown by Eq. (10) to efficiently perform static analyses with inertia relief and displacement-dependent loads. By comparing the partitions of Eq. (39) to the partitions of Eq. (10) defined by Eqs. (11), (12), and (13), it is clear the stated goal has been achieved. The upper-left partition, $[K_{yy}]$, is the original system stiffness matrix. This is analogous to the definition of $[K_{11}]$ shown in Eq. (10). The two lower partitions, $[T_{u_l y}]$ and $[T_{u_l u_l}]$, contain elements of the equations defining the u_l-set DOF displacements shown by Eq. (28). These partitions are analogous to the definitions of $[K_{21}]$ and $[K_{22}]$ shown in Eq. (10). The upper-right partition, $-[\rho_{yu_l}]$, defines all displacement-dependent loads applied to the system due to unit displacements of the u_l-set DOF. This is analogous to the definition of $[K_{12}]$ shown in Eq. (10).

The closed-form static equations with inertia relief and displacement-dependent loads shown by Eq. (39) are those implemented within MSC/NASTRAN Solution 91 via the new DMAP Alter.

Implementation

To include displacement-dependent loads during static analysis with inertia relief and solve the problem in closed-form, a new DMAP Alter has been developed for MSC/NASTRAN Solution 91. Prior to executing Solution 91 with the Alter, the analyst must generate required input. First, to facilitate the application of displacement-dependent loads, user-defined u_l-set DOF must be included in the structural model Bulk Data deck. Second, the displacement dependencies shown by Eq. (28) and displacement-dependent load relations shown in Eq. (29) must be generated and placed on DMIG Bulk Data cards. These relationships will be entered into the analysis via the Alter.

The alterations to Solution 91 for performing closed-form static analysis with inertia relief and displacement-dependent loads are as follows:

1. Read the displacement dependencies and displacement-dependent load relationship matrices into the analysis via DMIG Bulk Data cards.

2. Add the DMIG entered data to the structural stiffness matrix and reduce the combined g-set size stiffness matrix to the a-set size stiffness matrix. In effect, all DMIG entered data will form the matrix

$$[X_{aa}] = \begin{bmatrix} [0_{rr}] & [0_{ry}] & -[\bar{\rho}_{ru_l}] \\ [0_{yr}] & [0_{yy}] & -[\bar{\rho}_{yu_l}] \\ [T_{u_l r}] & [T_{u_l y}] & [T_{u_l u_l}] \end{bmatrix} \tag{40}$$

3. Account for the u_l-set DOF and reduce the a-set size stiffness matrix to the l-set size stiffness matrix:

$$[Y_{ll}] = \begin{bmatrix} [K_{yy}] & -[\bar{\rho}_{yu_l}] \\ [T_{u_l y}] & [T_{u_l u_l}] \end{bmatrix} \tag{41}$$

4. Generate the matrix of inertia loads due to unit displacements of the u_l-set DOF and add to the l-set size stiffness matrix:

$$[Z_{ll}] = \begin{bmatrix} [K_{yy}] & -[\rho_{yu_l}] \\ [T_{u_l y}] & [T_{u_l u_l}] \end{bmatrix} \tag{42}$$

   Note that $[Z_{ll}]$ shown by Eq. (42) is the system stiffness matrix shown on the left-hand-side of Eq. (39) and that the matrix is nonsymmetric.

5. Modify operations that assume symmetric matrices and solve Eq. (39).

In addition to these five alterations, the DMAP Alter has specialized operations for generating output specific to the analyses performed by the developers. One limitation of the Alter is that all DOF used in displacement dependencies and/or acted upon by displacement-dependent loads must be members of the residual structure a-set.

Numerical Example

The MSC/NASTRAN Solution 91 DMAP Alter for performing closed-form static analyses with inertia relief and displacement-dependent loads was developed to analyze a free-free ELV/spacecraft aerodynamic loads (static-aeroelastic) event. A typical static-aeroelastic analysis is illustrated in Figure 2. For this numerical example, only the lateral (+Y) component of the analysis will be discussed. The axial (+Z) component of the analysis can be performed in a similar manner, and the total results are simply the addition of the component analysis results. The goal of a lateral analysis is to calculate the lateral steady-state acceleration for a "trimmed" ELV/spacecraft system. A trimmed system is defined as a system acted upon by steady-state loads having only a lateral acceleration and zero in-plane rotational acceleration.

Referring to Figure 2, the free-free ELV/spacecraft system is assumed to be flying through the atmosphere at a given velocity. The velocity of the system is maintained by a gimballed engine thrust T. The velocity vector of the system is offset from the rigid-body centerline of the system by an angle α_o. This angle, the initial angle-of-attack, causes aerodynamic loads to act on the system. The resulting net lift force, located at the center-of-pressure which is offset from the system center-of-gravity, causes a moment. For the lateral component analysis, the engine gimbals to provide a lateral thrust L that will generate a moment about the coupled system center-of-gravity that counterbalances the moment generated by the aerodynamic loads. Hence, a non-zero lateral and a zero in-plane rotational steady-state acceleration are generated. However, the combined effects of the aerodynamic loads and lateral engine thrust causes the system to deform elastically. The system deformation causes local changes in the angle-of-attack that affects the aerodynamic loads. The changes in aerodynamic loads result in a new balancing lateral engine thrust being generated by the gimballed engine. Eventually, the system will reach the trimmed condition; whereby, moments generated by the aerodynamic loads and lateral engine thrust are balanced and the system is in its final deformed shape. In review, the free-free system equations are

$$[M_{aa}]\{\ddot{u}_a\} + [K_{aa}]\{u_a\} = \{\bar{P}_a\} + [\bar{\rho}_{aa}]\{u_a\} \tag{43}$$

where $\{\bar{P}_a\}$ are the initial aerodynamic loads and lateral engine thrust due to the initial angle-of-attack α_o, and $[\bar{\rho}_{aa}]$ is a matrix for transforming the system displacements into induced aerodynamic loads and lateral engine thrust. Note that Eq. (43) is of the same form as Eq. (25).

The main objective of an ELV/spacecraft static-aeroelastic analysis is to calculate the steady-state acceleration and final displacements of the trimmed system. The final displacements for the trimmed system can be solved for using the methodology presented in the previous section. Knowing the final system displacements, the final applied loads (aerodynamic and thrust) can be generated, and the lateral steady-state acceleration can be calculated as

$$\ddot{u}_{trimmed} = \frac{\{1\}^T\{P_a\} + \{1\}^T[\rho_{au_1}]\{u_{u_1}\}}{m_{lateral}} \tag{44}$$

where $m_{lateral}$ is the system lateral rigid-body mass.

The free-free ELV/spacecraft static-aeroelastic analysis illustrated in Figure 2 was first performed using an iterative solution technique and then performed using the new DMAP Alter. The iterative solution technique involved a series of Solution 91 analyses to solve Eq. (43) where output from one analysis was used to generate input for the next analysis. The analyses were performed until the difference in lateral steady-state accelerations shown by Eq. (44) between two successive analyses was below a specified tolerance. To meet the criterion, six iterations were performed. The second static-aeroelastic analysis was performed using Solution 91 and the new DMAP Alter to solve Eq. (43) in closed-form. The ratios of the lateral steady-state acceleration for each solution iteration divided by the lateral steady-state acceleration for the closed-form solution are shown in Figure 3. The final aerodynamic loads, final lateral engine thrust, and steady-state acceleration generated via the analyses are listed in Table 1.

Comparing the results generated via the two free-free ELV/spacecraft static-aeroelastic analyses shown in Figure 3 and Table 1, it is clear the new DMAP Alter for MSC/NASTRAN Solution 91 enables closed-form static analyses with inertia relief and displacement-dependent loads. From the data presented, the results generated using the iterative solution technique converge to the results generated using the closed-form solution technique.

Summary

A MSC/NASTRAN Solution 91 DMAP Alter has been written for performing closed-form static analyses with inertia relief and displacement-dependent loads. Through the combined use of DMAP and user-defined sets, static equations for a system model with displacement-dependent loads are generated in closed-form via linear displacement relationships. Special considerations are made for inertia relief effects due to the displacement-dependent loads. The Alter was written to replace iterative solution techniques typically used to solve a class of aerospace engineering problems. It has been shown via a numerical example that the new Alter allows for accurate solutions without the inefficiencies and added expenses typically associated with iterative solution methodologies.

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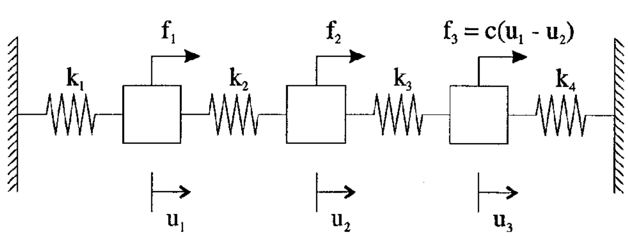

Figure 1.—Static system with displacement-dependent applied force.

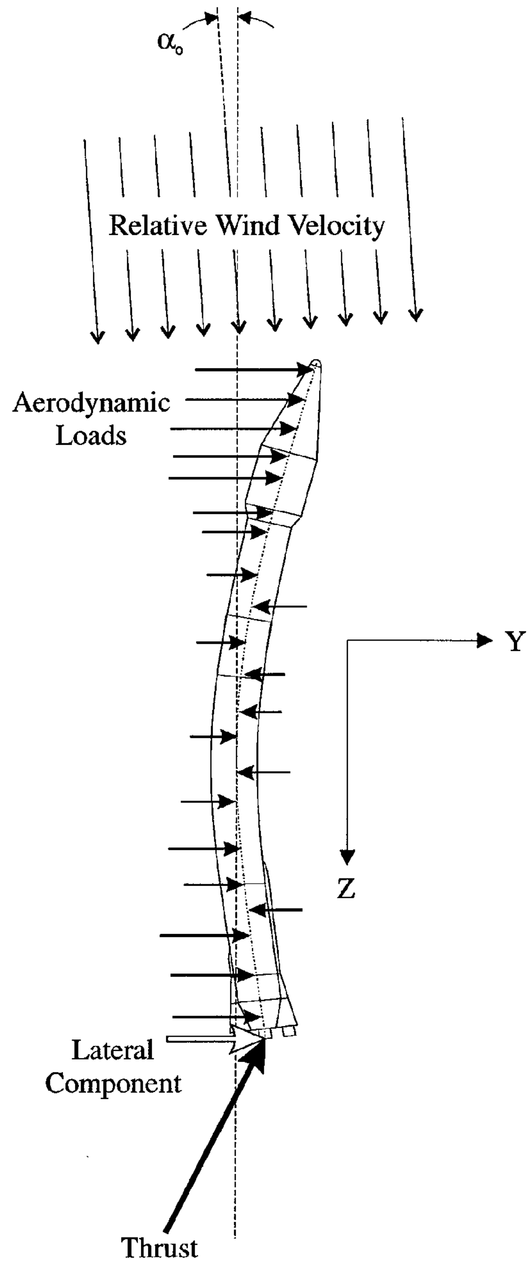

Figure 2.—Free-free ELV/spacecraft static-aeroelastic analysis.

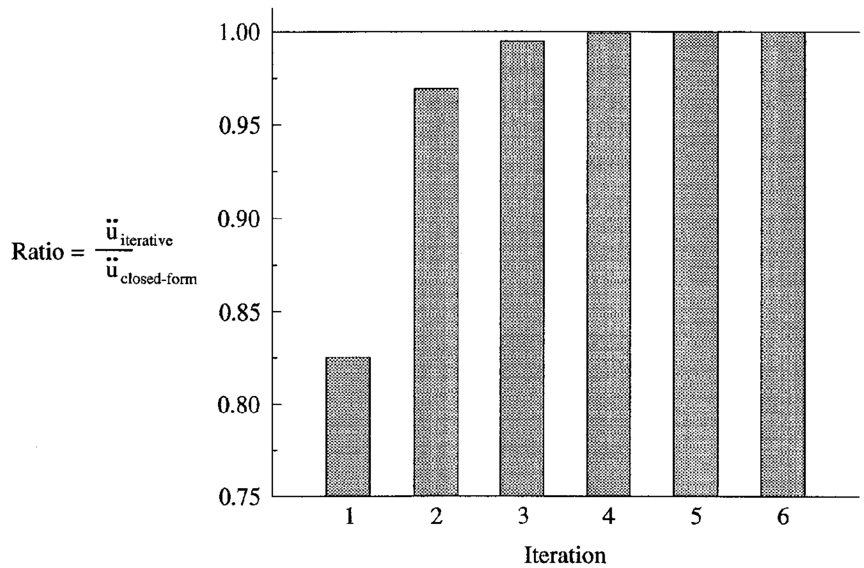

Figure 3.—Converging Lateral Steady-state Accelerations for ELV/spacecraft static-aeroelastic analyses.

Table 1. Final Results for Free-free ELV/spacecraft Static-aeroelastic Analyses

Quantity	Iterative Solution	Closed-form Solution	Ratio[a]
Final Aerodynamic Loads (lb)	25,684.306	25,684.269	1.00
Final Lateral Engine Thrust (lb)	39,888.319	39,888.361	1.00
Steady-state Acceleration (in/sec^2)	115.802	115.802	1.00

a - Ratio = (Closed-form solution result) / (Iterative solution result)