THERMAL CONDUCTION AND THERMAL CONVECTION AS A SINGLE THEORY SOLVED WITH FINITE ELEMENT ANALYSIS

By

Mircea Bocioaga
Romanian Aeronautical Enterprise
Brasov, ROMANIA

THERMAL CONDUCTION AND THERMAL CONVECTION AS A SINGLE THEORY SOLVED WITH FINITE ELEMENT METHOD

Mircea Bocioaga
Mathematician, Ph.D. engineer
Adress:
Str.Malaiesti nr. 14
2200 BRASOV
ROMANIA

ABSTRACT

This paper presents a theory in which thermal conduction and thermal convection is solved with a single equation. This equation is a generalised form of Fourier law. The paper presents a method, based on Ritz-Galerkin theory, for solving this equation. A main application for this equation could be the heat transfer study between a fluid flow and a solid body. The most important element is, that this theory is done without the convection theory and without the computation of a convection coefficient.

The domain in which the equation is solved is a finite element. The solution is a linear equation system where the unknown quantities are the temperature in the finite element nodes.

INTRODUCTION

It is known that thermal conduction is described with Fourier equation:

$$\frac{\partial t}{\partial \tau} = \frac{1}{\rho c}\left[\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial t}{\partial z}\right)\right] + \frac{q_v}{\rho c} \qquad (1)$$

t = temperature c = specific heat

τ = time q_v = heat generation per unit volume

$\lambda_{x,y,z}$ = thermal conductivity coefficient

ρ = density

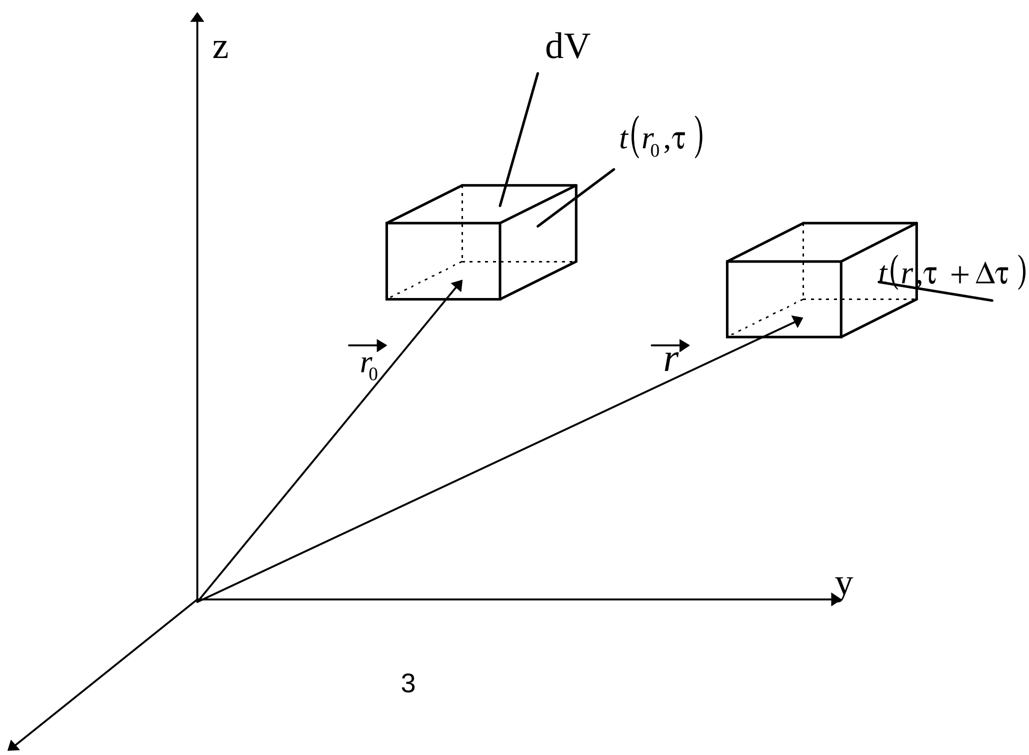
In equation (1) there are no terms to describe a change of place for the particles in the studied domain Ω. To describe a heat transfer associated with a change of place for the particles, we have to use the equation:

$$\frac{\partial t}{\partial \tau} + \frac{\partial t}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial t}{\partial y}\frac{\partial y}{\partial \tau} + \frac{\partial t}{\partial z}\frac{\partial z}{\partial \tau} = \frac{1}{\rho c}\left[\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial t}{\partial z}\right)\right] + \frac{q_v}{\rho c}$$

Here, the temperature is considered as a function t = t (r,τ) (fig. 1) and $\frac{\partial r}{\partial \tau} \neq 0$

$$\frac{\partial t}{\partial \tau} + w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} + w_z \frac{\partial t}{\partial z} = \frac{1}{\rho c}\left[\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial t}{\partial z}\right)\right] + \frac{q_v}{\rho c} \qquad (2)$$

$\vec{w}$ = speed of the particles



x

The supplementary term comes, obviously, when we have to make the thermal survey on a infinitesimal material element, and when we have to compute the expression:

$$\frac{dt}{d\tau} = \frac{dt\left(\vec{r},\tau\right)}{\partial\tau} = \frac{\partial t}{\partial\tau} + \frac{\partial t}{\partial \vec{r}}\frac{\partial \vec{r}}{\partial\tau}$$

A main application for this equation could be the heat transfer study between a fluid flow and a solid body. Looking at equation (2), the most important element is that the heat transfer study between a fluid flow and a solid body could be done without computing and using a convection coefficient.

SOLVING METHODOLOGY

Ritz-Galerkin method gives us the possibility to tackle a finite element analysis for solving the equation (2). I shall structure the presentation in two parts:

⇒ A first part in which I'll prove the existence of a functional equation on which is possible to apply Ritz-Galerkin method

⇒ A second part in which I shall apply the results on the generalised Fourier law equation (2).

The study will be done in the conditions of a steady state heat transfer $\left(\frac{\partial t}{\partial\tau} = 0\right)$

For the beginning I will write (2) as :

$$-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial t}{\partial x_i}\right) + \sum_{i=1}^{3} b_i\frac{\partial t}{\partial x_i} = f(x) \qquad (3)$$

$$x = \left(x_1, x_2, x_3\right) = \left(x, y, z\right) \in \Omega \subset R^3; a_i \in C^1(\Omega); b_i \in C(\Omega)$$

$$\sum_{i=1}^{3} a_i\xi_i^{\,2} \geq \gamma\left(\xi_1^{\,2} + \xi_2^{\,2} + \xi_3^{\,2}\right); \forall\xi \in R^3 \qquad (3')$$

With a Dirichlet condition:

$$t\,/_{\partial\Omega} = 0 \qquad (4)$$

where $\partial\Omega$ means the Ω domain frontier.

For analysing the problem (3) - (4) we shall use the Sobolev spaces $H^{2,1}(\Omega)$ and $H_O^{2,1}(\Omega)$

Using the definition from [1] we'll have:

$$H^{2,1}(\Omega) = \left\{u \in L^2(\Omega) \,/\, \exists D^\alpha u \in L^2(\Omega); \forall|\alpha| \leq 1\right\}$$

here $L^2(\Omega)$ is the multitude of function f:Ω→R where $\int_\Omega f^2 dx < \infty$

$D^{\alpha} u$ is the partial α derivative of a function

Writing down $C_0^{\infty}(\Omega)$ the multitude of the functions which have the support in Ω, we have:

$$\text{supp } u = \overline{\{ x \in \Omega \,/\, u(x) \neq 0 \}} \subset \Omega$$

The multitude $H_O^{2,1}(\Omega)$ is the closing of the multitude $C_0^{\infty}(\Omega)$. Is possible to associate scalar products to these multitudes. So, $H^{2,1}(\Omega)$ and $H_O^{2,1}(\Omega)$ becomes Hilbert spaces.

In [4] it is proved that is very simple to change a $u/_{\partial\Omega} = 0$ Dirichlet condition to a $u/_{\partial\Omega} = g$ (g ≠ 0) Dirichlet .

Now we can apply to the problem (3) - (4) the Ritz-Galerkin method. First, we have to take a function v∈ $H_O^{2,1}(\Omega)$, and to make the product:

$$-\sum_{i=1}^{3} v \frac{\partial}{\partial x_i}\left(a_i \frac{\partial t}{\partial x_i}\right) + \sum_{i=1}^{3} v b_i \frac{\partial t}{\partial x_i} = v f \qquad (5)$$

If we integrate (5) on the entire domain Ω, the result is:

$$-\int_{\Omega} \sum_{i=1}^{3} v \frac{\partial}{\partial x_i}\left(a_i \frac{\partial t}{\partial x_i}\right) dx + \int_{\Omega} \sum_{i=1}^{3} v b_i \frac{\partial t}{\partial x_i} dx = \int_{\Omega} v f dx \qquad (6)$$

For the first term from the left we can write:

$$-\int_{\Omega} v \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a_i \frac{\partial t}{\partial x_i}\right) dx = -\int_{\partial\Omega} v \sum_{i=1}^{3} a_i \frac{\partial t}{\partial x_i} \cos(N, x_i) d\sigma + \int_{\Omega} \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} \frac{\partial t}{\partial x_i} dx \qquad (7)$$

With condition (4), equation (7) becomes:

$$-\int_{\Omega} v \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a_i \frac{\partial t}{\partial x_i}\right) dx = \int_{\Omega} \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} \frac{\partial t}{\partial x_i} dx \qquad (8)$$

So, in the conditions of a problem with Dirichlet conditions, equation (6) becomes:

$$\int_{\Omega} \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} \frac{\partial t}{\partial x_i} dx + \int_{\Omega} \sum_{i=1}^{3} v b_i \frac{\partial t}{\partial x_i} dx = \int_{\Omega} v f \, dx \qquad (9)$$

As we can see equation (9) is a functional equation. Applying Ritz-Galerkin method, we search the solution as:

$$t_n = \sum_{k=1}^{n} c_k v_k ; c_k \in R \qquad (10)$$

in which the row $\{v_k\}$ forms a base in $H_O^{2,1}(\Omega)$ Hilbert space.

Using the functional equation (9), solving the linear system (11), we can find C_k constants from (10).

$$\int_\Omega \sum_{i=1}^{3} a_i \frac{\partial v_j}{\partial x_i}\frac{\partial t_n}{\partial x_i} dx + \int_\Omega \sum_{i=1}^{3} b_i v_j \frac{\partial t_n}{\partial x_i} dx = \int_\Omega v_j f dx \qquad (11)$$

j = 1,n

For any function f$\in L^2(\Omega)$ the linear system has a solution, and the solution is only one. This fact was proved by Prof. Kalik Carol in work [1] .

Here we have:

$$\gamma \|t_n\|_{1,0}^2 \le \int_\Omega \sum_i a_i \frac{\partial t_n}{\partial x_i}\frac{\partial t_n}{\partial x_i} dx + \int_\Omega \sum_i b_i t_n \frac{\partial t_n}{\partial x_i} dx = \int_\Omega t_n f dx \le \|f\|\|t_n\| \le \|f\| C \|t_n\|_{1,0} \quad (12)$$

Based on a demonstration from [1] results the fact that if we build the row $\{t_n\}$ with Ritz-Galerkin method, this row will converge in $H_O^{2,1}(\Omega)$ to the solution of the Dirichlet problem (3) - (4) .

If we consider a Neumann problem in a $H^{2,1}(\Omega)$ Hilbert space, the results will be the same.

Based on these results, it is possible now to consider the equation (2) written in steady state conditions. Taking any function v$\in H^{2,1}(\Omega)$, and making the product with (2), results:

$$\rho c\left(v w_x \frac{\partial t}{\partial x} + v w_y \frac{\partial t}{\partial y} + v w_z \frac{\partial t}{\partial z}\right) = v \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial t}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\lambda_y \frac{\partial t}{\partial y}\right) + v\frac{\partial}{\partial z}\left(\lambda_z \frac{\partial t}{\partial d}\right) + v q_v \quad (13)$$

or more:

$$\rho c \sum_{i=1}^{3} v w_{xi} \frac{\partial t}{\partial x_i} - \sum_{i=1}^{3} v \frac{\partial}{\partial x_i}\left(\lambda_{xi}\frac{\partial t}{\partial x_i}\right) = q v$$

(14)

Here I used the convention $x_1 = x; x_2 = y; x_3 = z; x = (x_1, x_2, x_3)$ (15)

Putting (14) under the integral sign on the whole domain Ω, yields:

$$\rho c \int_\Omega \sum_{i=1}^{3} v w_{xi} \frac{\partial t}{\partial x_i} dx - \int_\Omega \sum_{i=1}^{3} v \frac{\partial}{\partial x_i}\left(\lambda_{xi} \frac{\partial t}{\partial x_i}\right) dx = \int_\Omega v q_v dx$$

(16)

In (16) we can compute the integral from parts:

$$-\int_{\Omega}\sum_{i=1}^{3} v\frac{\partial}{\partial x_i}\left(\lambda_{xi}\frac{\partial t}{\partial x_i}\right)dx = -\int_{\partial\Omega} v\sum_{i=1}^{3}\lambda_{xi}\frac{\partial t}{\partial x_i}\cos\left(N, x_i\right)d\sigma + \int_{\Omega}\sum_{i=1}^{3}\lambda_{xi}\frac{\partial v}{\partial x_i}\frac{\partial t}{\partial x_i}dx \tag{17}$$

To solve equation (14), we have to put now some boundary conditions. I'll consider some imposed heat flux conditions on the frontier of the Ω domain (a Neumann problem)

- for the imposed heat flux zones:

$$q = \lambda_x\frac{\partial t}{\partial x}n_x + \lambda_y\frac{\partial t}{\partial y}n_y + \lambda_z\frac{\partial t}{\partial z}n_z = \sum_{i=1}^{3}\lambda_{xi}\frac{\partial t}{\partial x_i}\cos\left(N, x_i\right) \tag{18}$$

- for the heat convection flux zones:

$$\alpha\left(t - t_E\right) = \sum_{i=1}^{3}\lambda_{xi}\frac{\partial t}{\partial x_i}\cos\left(N, x_i\right) \tag{19}$$

where $n_{xi} = \cos\left(N, x_i\right)$ are the components of the normal versor on the surface.

Using (18) and (19) is possible to rewrite (17):

$$-\int_{\Omega}\sum_{i=1}^{3} v\frac{\partial}{\partial x_i}\left(\lambda_{xi}\frac{\partial t}{\partial x_i}\right)dx = -\int_{\partial\Omega_1} qv\,d\sigma_1 + \int_{\partial\Omega_2}\alpha v\left(t - t_E\right)d\sigma_2 + \int_{\Omega}\sum_{i=1}^{3}\lambda_{xi}\frac{\partial v}{\partial x_i}\frac{\partial t}{\partial x_i}dx \tag{20}$$

where: t_E - ambient temperature

α - convection coefficient

$\partial\Omega_1$ - part of the Ω domain frontier with a imposed heat flux

$\partial\Omega_2$ - part of the Ω domain frontier where is a convection heat exchange

Equation (16) becomes now:

$$\rho c\int_{\Omega} v\sum_{i=1}^{3} w_{xi}\frac{\partial t}{\partial x_i}dx - \int_{\partial\Omega_1} qv\,d\sigma_1 + \int_{\partial\Omega_2}\alpha v\left(t - t_E\right)d\sigma_2 + \int_{\Omega}\sum_{i=1}^{3}\lambda_{xi}\frac{\partial v}{\partial x_i}\frac{\partial t}{\partial x_i}dx = \int_{\Omega} vq_v\,dx \tag{21}$$

At this moment is possible to apply on (21) Ritz-Galerkin method. I'll search the temperature as:

$$t_n = \sum_{k=1}^{n} c_k v_k ; c_k \in \overline{R}$$

(22)

where V_k ; k=1,n are linear independent functions in $H^{2,1}(\Omega)$ [1] .

To determine these functions I'll appeal to the finite element theory. In this theory are used some interpolation functions [2], [3], [4]

With these interpolation functions, the temperature in the interior of the finite element is written as:

$$t = \sum_{k=1}^{n} t_k v_k = [v_k]_{element} \{t_k\}_{element}$$

(23)

Here t_k k = 1 , n are the temperature values in the nodes. The functions V_k are polynomial functions and belongs to Sobolev space $H^{2,1}(\Omega)$. Because they are forming a base in the space $H^{2,1}(\Omega)$ these functions are linear independent. These functions are forming a base because any temperature from the finite element can be written like a linear combination with them. Now we can replace (23) in (21). The result is:

$$\rho c \int_{\Omega} v_j \sum_{i=1}^{3} w_{xi} \sum_{k=1}^{n} t_k \frac{\partial v_k}{\partial x_i} dx - \int_{\partial\Omega_1} q v_j d\sigma_1 + \int_{\partial\Omega_2} \alpha v_j \left(\sum_{k=1}^{n} t_k v_k - t_E \right) d\sigma_2 + \int_{\Omega} \sum_{i=1}^{3} \lambda_{xi} \frac{\partial v_j}{\partial x_i} \sum_{k=1}^{n} t_k \frac{\partial v_k}{\partial x_i} dx = \int_{\Omega} v_j q_v dx$$

$j = \overline{1,n}$

(24)

In (24) "n" represents the number of nodes from the finite element.

This linear system of equations has n equations and the unknown quantities are t_k ; k =1,n , the temperature in the finite element nodes.

Even if this equations solves the conductive and convective heat transfer in the Ω domain (the finite element domain), it is possible to consider as a classical assumption, a face heat convection (α - convection coefficient) or a face heat flux (q) on the domain (finite element) frontier.

The system (24) can be used to compute at a finite element level a conductive-convective heat transfer process. Due his form, this system can be brought at the form:

$$[\ K\ ]_{element}\ \{\ t\}_{element} = \{\ f\ \}$$

In the end it is possible to assemble these systems (written for only one finite element) for the whole domain.

To test this theory I made little computer programm. I took four plane finite elements. The input data is:

	ELEMENT 1	ELEMENT 2	ELEMENT 3	ELEMENT 4
node 1 coordinates [m]	0 ; 0	0 ; 0.01	0 ; 0.015	0 ; 0.02
node 2 coordinates [m]	0.05 ; 0	0.05 ; 0.01	0.05 ; 0.015	0.05 ; 0.02
node 3 coordinates [m]	0.05 ; 0.01	0.05 ; 0.015	0.05 ; 0.02	0.05 ; 0.03
node 4 coordinates [m]	0 ; 0.01	0 ; 0.015	0 ; 0.02	0 ; 0.03
λ_x,λ_y [W/mK]	105 ; 105	0.136 ; 0.136	0.136 ; 0.136	0.136 ; 0.136
speed [m/s] w_x,w_y	0 ; 0	0.2 ; 0.2	0.2 ; 0.2	0.2 ; 0.2
mass density [kg/m^3]	8900	900	900	900
specific heat [J/Kg K]	386	2000	2000	2000
α [W/m^2 K]	90	0	0	0
ambient temperature [^oC]	20	0	0	0

The results are:

90 (imposed)	89.9
Element 4	
50.1	
50.1	
Element 3	
31.5	
31.5	
Element 2	

If I consider the speed = 0 results

90 (imposed)	85.3
Element 4	
57.79	
54.6	
Element 3	
41.5	39.5
Element 2	

22	22
Element 1	
22	22

Ambient temperature 20 ; $\alpha = 90$

25.4	24.5
Element 1	
25	24.5

Ambient temperature 20 ; $\alpha = 90$

CONCLUSIONS

This theory may be the basis for a new MSC/NASTRAN product.

As we can see, to solve the problem it is necessary to have or to know the speed field in the whole domain. For this reason, I think that this theory may be a link between MSC/NASTRAN THERMAL and a soft which, based on Navier-Stokes equations, gives the domain speed field. For example MSC/AEROELASTICITY.

This theory, and a virtual new MSC product, may be useful for that part of user community working in aviation.

This theory is possible to be considered as a generalisation for the classical thermal analysis. The results are logic and is possible to modelate a limit thermal layer and a heat exchange between a solid body and a fluid flow.

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